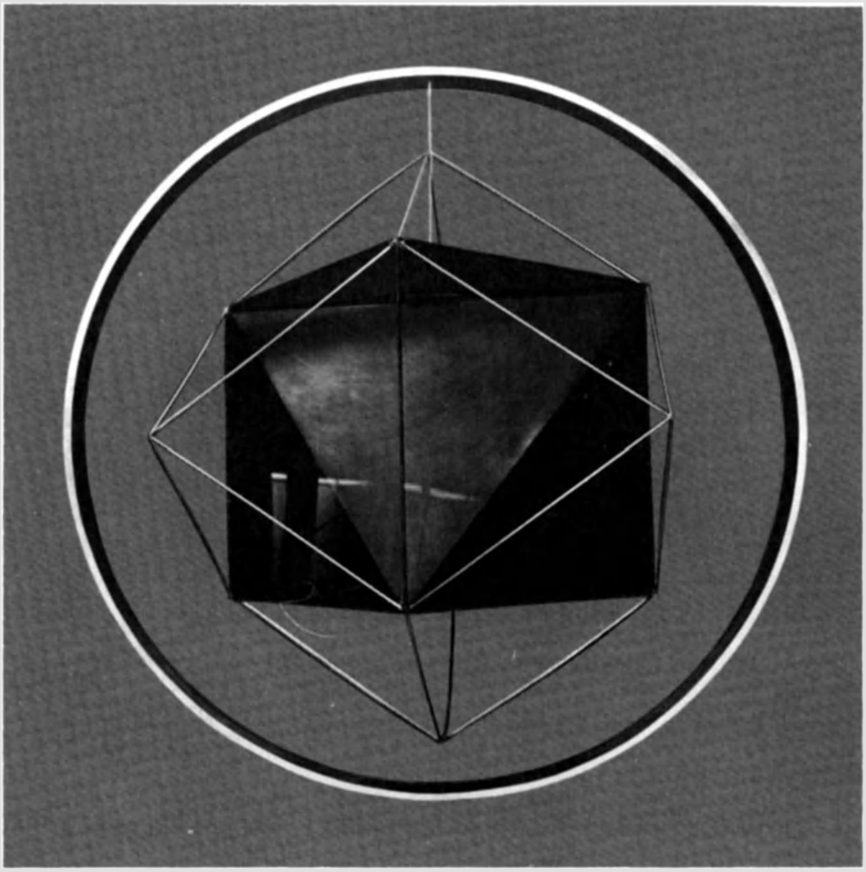
Space Structures

Their Harmony and Counterpoint



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Space Structures

Their Harmony and Counterpoint

Arthur L. Loeb

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Harvard University

With a Foreword by Cyril Stanley Smith

A Pro Scientia Viva Title

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To the Philomorphs,
Fellows in the Search for Order

Design Science Collection

Series Editor
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Foreword

Good wine needs no bush, and a book on structure by Arthur Loeb can hardly be improved by a foreword. Yet it may be pointed out that this book has a simple fundamental quality of viewpoint and treatment that gives it an unusually widespread applicability. It is not just for mathematicians and crystallographers.

The author and I, respectively a mathematician and a metallurgist, have often met as members of a group of people who call themselves the Philomorphs: biologists, artists, crystallographers, architects, sociologists and others—brought together by a common interest in the underlying patterns of interaction between things. Some awareness of structure lies immediately beneath the surface in virtually every field of abstract knowledge or purposeful activity, as well as in games, entertainment and the arts. In many fields the whole purpose is the finding of significant structure and the understanding and control of structural change. Structural terminology appears, often unnoticed, in everyday conversation as well as in learned treatises, although in most cases the interest is not in the structure itself, but rather in the qualities or properties to which structural interactions give rise.

Most considerations of structure in the past have overemphasized polyhedra, unit cells and symmetry. Arthur Loeb emphasizes valency, the number of connections pertaining to each structural feature. The concept of valency, originally simply a way of understanding the fixed compositions of chemical compounds, is here used as a basic feature in more complicated structures. Valencies are combined with dimensionality, and the inescapable interlock is exploited between the valence of points, lines, polygons and cells in aggregates: this touches on the very nature of things. The concept of statistical symmetry which Loeb develops is particularly important, for it emphasizes the limitations in seemingly random aggregates and permits general statements of which the crystallographer's symmetries are only special cases.

The reductionist and holistic approaches to the world have been at war with each other since the times of the Greek philosophers and before. In nature, parts clearly do fit together into real structures, and the parts are affected by their environment. The problem is one of understanding. The mystery that remains lies largely in the nature of structural hierarchy, for the human mind can examine nature on many different scales sequentially but not simultaneously. Arthur Loeb's monograph is a fundamental one, but one can sense a development from the relations between his zero- and three-dimensional cells to the far more complex world of organisms and concepts. It is structure that makes the difference between a cornfield and a cake, between an aggregate of cells and a human being, between a random group of human beings and a society. We can perceive anything only when we perceive its structure, and we think by structural analogy and comparison.

Several books have been published showing the beauty of form in nature. This one has the beauty of a work of art, but it grows out of rigorous mathematics and from the simplest of bases—dimensionality, extent and valency. Structure in the decorative arts depends on the same repetition and variation of space-filling relationships that are discussed here. The architect or builder uses them as he turns bricks into walls, walls to rooms, and rooms to buildings. I myself as a metallurgist see its immediate applicability both to the symmetry of most alloy crystals and to the non-symmetrical space-filling aggregates of crystalline grains and other microconstituents. The prevalence of pentagons in nature is no mystery—it is inevitable in a random aggregate of undifferentiated cells of any kind, for it is simply a result of undirectional contiguity and economy of interface.

Underlying all this is the Euler-Schlaefli relation, a statement of dimensional connectedness so fundamentally true that it is inescapable and therefore explains nothing beyond existence. Here we move from this to a consideration of many types of special units, always within the general framework. It is good fundamental mathematics applicable to anything, but the viewpoint is such that it leads to the study of complexity without destroying individuality.

CYRIL STANLEY SMITH

Pretext

"Anyone who has read this book will realize that the use of the term *preface* would imply the existence as well of a prevertex, a preedge, and a precell. In order not to enrich the English language unduly, I am making use of a *pretext* here." This is how I started the first edition of *Space Structures;* of course I was being *face*tious, being well aware that the word *Preface* has its origin in the word *Prefatio,* and is not related to the word *Face.*

The reason why I started with a pun is, however, to stress that this is a book about language. In a broad sense Design Science is the grammar of a language of images rather than of words. Modern communication techniques enable us to transmit and reconstitute images without the need of knowing a specific verbal sequential language such as the Morse code, or Hungarian. International traffic signs use international image symbols which are not specific to any particular verbal language. An image language differs from a verbal one in that the latter uses a linear string of symbols, whereas the former is multidimensional.

The terms *harmony* and *counterpoint* are used by musicologists to describe quantitatively the interrelationships between structural components of a musical composition. The science of the structural analysis of a musical pattern has been practiced for centuries, and has reached a degree of sophistication and a range of use that has no equal in design science. It is my hope that our brethren in the tonal arts will consider my use of their favorite terms in the subtitle of this book a tribute and a gesture of admiration.

Perception is a complex process. Our senses record; they are analogous to audio or video devices. We cannot, however, claim that such devices perceive. Perception involves more than meets the eye: it involves processing and organization of recorded data. When we name an object, we actually name a concept: such words as *octahedron, collage, tessellation, dome,* each designate a wide variety of objects sharing certain characteristics. When we devise ways of transforming an octahedron, or determine whether a given shape will tessellate the plane, we make use of these characteristics, which are aspects of the grammar of structure.

The *Design Science Collection* concerns itself with this grammar. The basic parameters of structure such as symmetry, connectivity, stability, shape, color, size, recur throughout these volumes. Their interactions are complex; together they generate such concepts as Fuller's and Snelson's tensegrity, Lois Swirnoff's modulation of surface through color, self reference in the work of M. C. Escher, or the synergetic stability of ganged unstable polyhedra. All of these occupy some of the professionals concerned with the complexity of the space in which we live, and which we shape. These professionals do not all speak the same language, and in planning this book I faced this dilemma: should I attempt to cover special applications of interest to certain specialists, or should I limit myself to topics of common interest? To put it in the language of logicians: should I address myself to the *union* or the *intersection* of these design science specialties? Such an intersection would be small at most, and a book addressing only those topics common to all relevant disciplines would have limited appeal.

Within our culture modern natural sciences and mathematics have become less accessible because of the specialized skills required to become conversant with them. Nevertheless, in piling discovery upon discovery, we do not usually follow the most direct path to a concept, and if we were to retrace our steps we might well recognize that not all the byways by which such a concept was developed historically, are actually prerequired for understanding the concept. Although such retracing requires time and effort, it will open up channels of communication that may enrich our culture.

In *Space Structures* I have identified concepts basic to space structures, and expressed these in a common language. In doing so, I ran the risk of irritating the mathematician by explaining a summation sign that might be unknown to the architect, and I might discuss degrees of freedom for a dome structure in terms that a molecular spectroscopist considered her private domain. There have been places where I felt that mathematical rigor had to be sacrificed when common sense and intuition left no doubt about the truth of an experimental result, and a rigorous proof would alienate too many readers. We are fortunate that mathematics exists to confirm what we consider virtually certain, and Design Scientists should be able to accept such confirmation on faith from the mathematician without necessarily following each step. The Philomorphs, to whom this volume was originally dedicated, and who continue to conduct lively monthly meetings, have borne such interdisciplinary translation with fortitude and courtesy, and have benefited from them. Thus I have come to take for granted a certain forbearance on the part of my widely varied audience.

Space Structures was initially published in 1976. We owe thanks to the original publisher, Addison Wesley Publishing Company, Inc. and to Lore Henlein, at the time of the original publication in charge of its Advanced Book Program, for their assistance in reprinting the volume in the *Design Science Collection*. Since this original publication the author has been teaching Design Science in Harvard's Department of Visual and Environmental Studies in the Carpenter Center for the Visual Arts, a department devoted to turning out students articulate in images, much as a language department teaches reading and expressing oneself in words. The book has been used continuously as a text in VES 176 (Synergetics, the Structure of Ordered Space), and Studio Arts 125b (Design Science Workshop, Three-dimensional).

In these courses I have worked to overcome visual illiteracy and mathematics anxiety, two serious and related problems. Visual illiteracy affects our man-made environment and its relation to our natural ecology. Mathematics anxiety deprives those afflicted of access to the grammar needed to express oneself spatially. The visual clutter of our built environment and the sterility of ultra-functionalist design reflect the intellectual poverty of our environmental design. Only a proper understanding of the constraints imposed by the properties of our space and of the rich repertoire permitted within these constraints allow the achievement of a balanced disciplined freedom.

It is a pleasure here to acknowledge the influence of my students, whose enthusiasm and whose questions helped shape this volume. A few have made contributions that are specifically acknowledged in appropriate places. Since the original publication my course assistants, Holly Alderman, William L. Hall, Jack Gray,

William Varney and P. Frances Randolph, the latter two former students as well, have rendered invaluable service in interpreting the material in this and the other volumes in the *Design Science Collection*. Special thanks are due to William Hall and Jack Gray for organizing and documenting our Studio library and references. The bibliography appended to the present edition would not have been possible without their assistance.

I hope that to some alumni of my courses this monograph represents a recollection of what to me were valuable and enjoyable hours, and that it will give pleasure to future generations in developing their visual literacy and improving their visual environment.

ARTHUR L. LOEB
Cambridge, Massachusetts,
January 1991

Introduction

The subdivision of space and the quantitative expression of spatial order are a concern to all who deal with spatial patterns, whether with crystals whose interacting elements are but angstroms apart, or with cities extending over many kilometers.

Space is not a passive vacuum, but has properties that impose powerful constraints on any structure that inhabits it. These constraints are independent of specific interactive forces, hence geometrical in nature. This book deals with the nature of these geometrical constraints, as well as with quantitative means of expressing them.

A number of beautiful handbooks and review articles have appeared recently, enumerating and illustrating a wealth of polyhedral structures, and rich in numerical data about these structures. A selection of these books and articles is listed in the bibliographical section at the end of this book. One aim of the present volume is to present fundamental principles underlying this variety of structures, to show their family relationships and how they may be transformed into one another—in short, to present a system relating them. We might speak therefore of the *structure of structures*.

It is appropriate here to define *structure*. Let us assume that we know what an *array* is: simply a collection of entities. A *pattern* is an *ordered* array; the different entities in the array bear a well-defined relation to each other. The set of relationships between the entities in a pattern is called the *structure* of the pattern.

The pattern does not need to be a visual one. It could be an organization, such as a corporation. The structure of the corporation expresses reporting relations, interactions between subsidiaries, financial accountability, etc. The structure of a polyhedron is expressed by its symmetries, by the numbers of vertices, edges, and faces it contains, by the numbers of edges and faces that meet at each vertex, etc. A study of *structure* therefore implies a study of *relations.* And when I speak of a *structure of structures,* I mean the set of relationships between different structures.

We shall begin our study by interconnecting an array of randomly chosen points, and we shall discover that, at least statistically, there exists law and order even in such random systems. This starting point is thus at the opposite end of the scale from that in *Color and Symmetry.*[1] In a study of symmetry one deals with patterns in which many angles are exactly equal to each other and many distances exactly equal to each other. Here we shall, throughout most of the book, be totally unconcerned with exact distances and angles. Rather, we shall deal with *numbers* of interconnections, *numbers* of vertices, faces, edges. As we proceed, we shall follow a program of increasing order, arriving eventually at the most symmetrical structures.

Numbers rather than *distances* are our principal parameters. We tend to think of numbers as a linear array, related to each other by a single pair of relationships: "larger than" and "smaller than." We know, however, that *imaginary* numbers are not so simply related, and that the concept of imaginary numbers and variables has been invaluable in understanding, for instance, electrical communication systems. Throughout history, technical progress has paralleled the expansion of symbolic languages. The language of structure to be explored here deals with numbers as three-dimensional entities: we shall think of a number as a distribution in space of structural elements.

[1]Arthur L. Loeb: *Color and Symmetry* (Wiley, New York, 1971).

1

The Parameters of Structure

We shall express structure in terms of three kinds of parameters:

1. Dimensionality
2. Valency
3. Extent

Each of these requires some elaboration. *Dimensionality* refers to numbers of degrees of freedom. A point confined inside a cube may move in *three* mutually perpendicular independent directions; any other motion of the point is expressible as a combination of motions in these three directions. The point has *three* degrees of freedom within the cube. Therefore the space inside the cube is *three-dimensional.* A point confined to the *surface* of a cube has only *two* independent directions of motion—i.e., *two* degrees of freedom: the surface of the cube is *two-dimensional,* even though not flat. Similarly, the space inside a sphere is three-dimensional, because a point confined to the inside of the sphere has three degrees of freedom. The surface of the sphere, however, while not flat, is *two-dimensional,* because a point confined to the surface of a sphere has *two* degrees of freedom. On the surface of the planet Earth these two degrees of freedom are commonly expressed as longitude and latitude. On a perfect sphere the distance from its center to any point on its surface is the same. Earth is not a perfect sphere: the poles are closer to its center than is any other point, while every point of the equator appears to be further from its center than any other point. Nevertheless, although the distance from the center of the Earth to any point on its surface varies with the latitude of that point, this distance is fixed for every given latittude, hence does not represent a third degree of freedom. In this discussion we have ignored the effect of altitude of mountains and depth of oceans. Nevertheless, what we have said holds true regardless of pimples or dimples on the surface: whenever we discuss a *surface* there are only two degrees of freedom.

Surfaces may enclose a finite space, as in the case of the cube and sphere, or they may extend indefinitely, as would a plane. The closure of the surface may occur in various ways: the sphere is one example. One might, as a second example, take a cylindrical surface, and bend the ends around until they meet to form a doughnut. The sphere is called singly connected, the doughnut doubly connected. The explanation is as follows: Imagine two points inside a sphere, connected by two rubber bands. If these rubber bands are allowed to contract, they will do so, and coincide along a line joining the points. On the other hand, imagine two points inside a doughnut, also connected by two rubber bands. One of these bands goes in one direction around the doughnut from one point to the other, whereas the other starts in the opposite direction, and eventually meets the other band at the second point (cf. Fig. 1-1). In this instance, shrinking the bands will *not* bring them into alignment, because the doughnut hole keeps them apart. Hence, if a multiplicity of rubber bands can be used to connect two points inside a closed surface in such a way that shrinking the bands will not bring them into alignment, then the surface is multiply connected. The space between two concentric spheres is singly connected, because two rubber bands con-

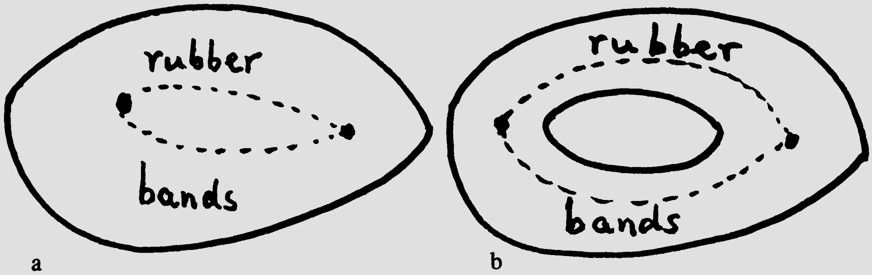


FIGURE 1-1a and 1-1b Singly (a) and doubly (b) connected surfaces necting a pair of points can be brought into coincidence, even if not along a straight line connecting the points.

We shall confine ourselves to singly connected surfaces, not because multiply connected surfaces are not interesting, but because there is a great deal to be said even within these confines, and because the same principles to be discussed here may be extended to the more complex surfaces once they are understood for the simpler ones.

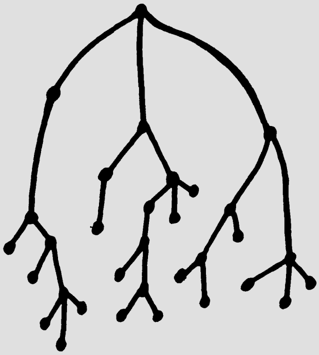
A point confined to move along a curve has a single degree of freedom: it can move forward and backward only. These motions are *not* independent, because a forward motion can always be balanced by an appropriate backward motion. A curve is therefore *one-dimensional.* When two surfaces, which have dimensionality *two,* intersect, they do so along a curve, which has dimensionality *one.* Analogously, curves that intersect do so at a *point,* which has dimensionality *zero,* having no degree of freedom at all.

In our three-dimensional space we have elements of dimensionality 0, 1, 2, and 3. Those of dimensionality 0 we call *vertices,* those of dimensionality 1 are called *edges,* those of dimensionality 2 *faces,* and those of dimensionality 3 *cells.* We just saw that curves intersect in points, having dimensionality 0. These points are the *vertices* of our structures. Any portion of a curve that joins two vertices but does not itself contain a vertex except the two terminal ones is called an *edge.* An edge or set of edges that encircle a portion of a surface is said to enclose a *face* if that portion of the surface does not contain any edge in addition to the ones encircling it. Finally, a *cell* is a portion of space that is entirely enclosed by faces and edges, and does not internally contain any faces or edges. It must be emphasized that edges need not be, and generally are not, straight lines, and that faces are not necessarily planes.

Our structures are then systems of interconnected elements of dimensionalities 0, 1, 2, and 3. We shall see that the numbers of each dimensionality may not all be fixed arbitrarily, but are interrelated. The structures that we are about to consider are discrete: each vertex is joined to a finite number of other vertices by a finite number of edges; each edge is joined to a finite number of other edges by a finite number of faces. Each face separates exactly *two* cells, and each edge joins exactly *two* vertices. The number of elements of a given dimensionality that meet at an element of different dimensionality is called a *valency;* valencies will be defined and discussed in greater detail in the next chapter.

Having discussed the parameters *dimensionality* and *valency,* we are left with the parameters of *extent.* The parameters of extent are the most familiar. They are simply *length, area,* and *volume* for elements of respective dimensionalities 1, 2, and 3. In my *Contributions* to R. Buckminster Fuller's *Synergetics,*[1] I have pointed out that the commonly assumed relations between extent parameters of elements of different dimensionalities are special cases suitable in special frameworks only. Parameters of extent become important only toward the end of this book, where they will be (re)-examined accordingly.

[1] R. Buckminster Fuller, with E. J. Applewhite and A. L. Loeb: *Synergetics* (Macmillan, New York, 1975), pp. 832-836.



2

Valencies

Figure 2-1 represents the number *zero.* There is nothing: every point is equivalent to every other one. The significance of this figure is appreciated only by comparison with Fig. 2-2, which represents the number 1. The appearance of a single unique point at once establishes a center of reference. The extension to Fig. 2-3, the number 2, is stupendous: instead of a single central point we have now *two* vertices, between which there can be a relation. This relation may be one of equivalence, or one of opposition or polarization. Whichever it is does not matter: important is that *two* are necessary to establish a *relation,* and that *one* was necessary for a center of reference.

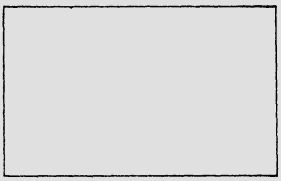


FIGURE 2-1
The number zero

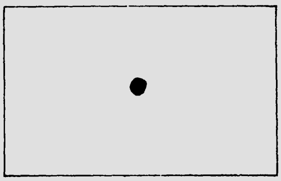


FIGURE 2-2
The number *1*

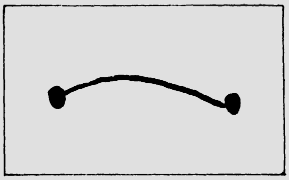


FIGURE 2-3
The number *2*

If we call the two entities representing the number 2 *A* and *B,* then the fact that *A* bears a relation to *B* implies that *B* also bears a relation to *A*. These two relations are not necessarily the same: the relation "*A* is married to *B*" implies "*B* is married to *A,*" but "*A* is the son of *B*" implies "*B* is a parent of *A*." In the first instance the relation of *A* to *B* is identical to the relation of *B* to *A;* in the second it is not. The relation "A is the wife of B" implies "B is the husband of A" as well as "A is married to B." Thus we see that, as long as there is a single relationship linking A and B, there may be many others. However, we can denote the existence of a relationship or relationships between two entities as a single edge joining two vertices, as was done in Fig. 2-3. This figure is sometimes called a *graph,* being a graphical representation of the structure of the number 2. If it is important to designate that the relation $A \rightarrow B$ is not the same as the relation $B \rightarrow A$ (e.g., "A is the son of B"), an arrow may be placed along the edge in the graph. In this case we speak of a *directed graph.* If the relationship of A to B is the same as that of B to A (e.g., "A is married to B"), or if the exact nature of the relationship is immaterial, we have an *undirected graph.* We shall here be concerned with undirected graphs, in which vertices represent entities, and edges (the existence of) relations between them.

Therefore, we see that the single edge connecting two vertices implies a highly complex set of relationships, a complex multidimensional system. This is the significance of the number *2:* it is the minimal number of entities between which a system of relationships may exist. One might conceivably have a single entity or vertex, and an edge looping from that vertex back to itself. However, such self-relationships are rather trivial, and do not really enter into a structural system, although they could be included without loss of generality. We shall not explicitly consider them here, though.

Figure 2-4 introduces further new principles. It represents the number 3. Now each entity bears a (generally different) relationship to each of the other two entities. The graph shows clearly that there are three edges–i.e., three *independent* relations. Suppose that A is the father of C, and B is the mother of C. Implied are that C is a child of A, and C is a child of B; these relationships are not independent of the original supposition. However, the relation between A and B is not defined by the supposition: A may be married to B, but A and B may be divorced. The relation of A to B is therefore, if it exists at all, an independent relation. The structure of Fig 2-4 is a closed one, in which each of A, B, and C bears an independent relationship to the other two. This closure relationship is represented in the graph by the face enclosed by the three edges (cf. "tree graphs" discussed below).

I have discussed these types of relationships in some detail because the representation of the numbers 2 and 3 in Figs. 2-3 and 2-4, respectively, may at first glance have appeared trivial. The importance of graph representations of structures becomes clear, however, when we proceed to Fig. 2-5, representing the number 4. Suppose that A is the father of C and D, and B is the mother of C. Without a graph it would be very difficult to evaluate the number of independent relations that might exist. A may or may not be married to B; B may or may not be the mother of D; C and D may be brother and sister, brothers, sisters, half-brother and half-sister, etc. However, we

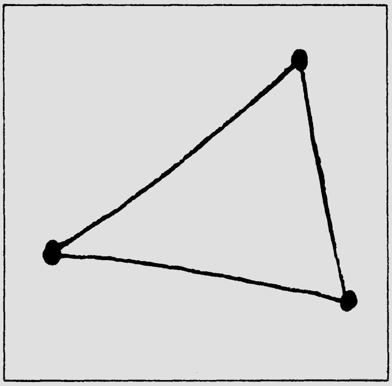


FIGURE 2-4 The number *3*

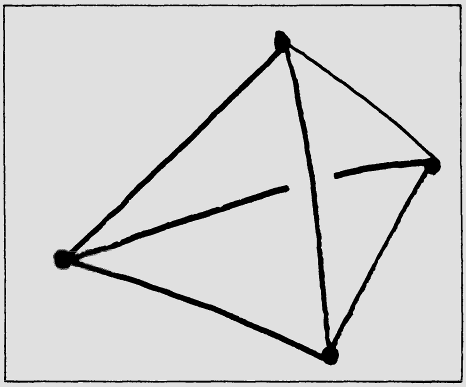


FIGURE 2-5 The number *4*

see clearly that there are six independent relationships. The *four* vertices are connected by *six* edges. There are four different closed circuits around the four faces, and the four faces enclose a single cell. Or, putting the last statement slightly differently, the four faces interconnect so as to create a singly connected surface that divides the three-dimensional space into an enclosed cell and all the space outside it. Similarly, the closed circuit of Fig. 2-4 divides a two-dimensional surface into an enclosed face and all the surface outside it. This concept of dividing the surface in two by a closed circuit and space in two by means of a closed surface will shortly prove to be far from trivial.

Figure 2-6 shows a graph in which one entity is related to several others, but the relation between these others among themselves is left dangling. Such a graph is called a tree graph. We exclude tree graphs from consideration: in our structures all circuits are closed, and all faces join into closed, singly connected surfaces. This means that *at least* two edges join at every vertex, and *at least* two faces meet at every edge. (Closure postulate)

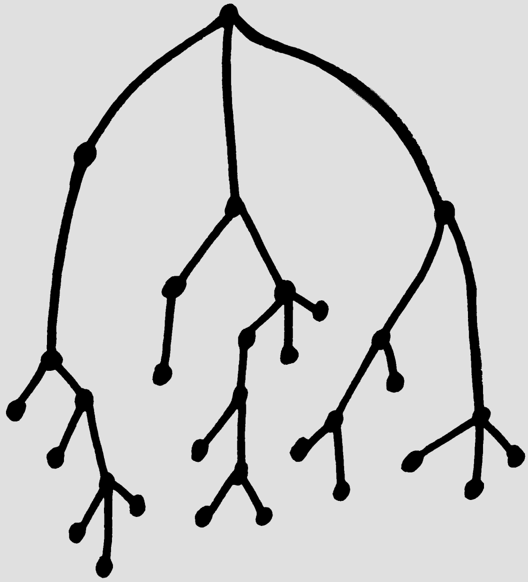


FIGURE 2-6 A tree graph

What does this manifesto do to our Figs. 2-3 and 2-4? It excludes them from consideration unless we look at them in a very special manner. We tend to think of the page on which they are printed as flat and infinitely extended. However, this image is at variance with our resolution to have all surfaces singly connected. We must therefore imagine these figures as printed on a very large sphere: the two points are then joined by *two* edges which together gird the sphere—one very short, the other very long (but their *extents* do not concern us here!). This closed circuit made by the two edges then divides the sphere surface into *two* faces. Analogously, the triangular circuit of Fig. 2-4 divides the huge sphere into two faces—one small, the other large. Analogously also, this sphere divides space into *two* cells: *inside* and *outside.* The reason for this apparent artifice will become obvious when we consider more complex structures, making constant use of the closure hypothesis. When the general results derived from this hypothesis are applied to the rather more trivial structures having two or three vertices, there would be apparent contradictions unless these simple structures are examined from the same point of view. If we had not been raised solely on Euclidean principles, the artifices would not appear to be so artificial.

More palatable, however, is another way of looking at the closure hypothesis. In three-dimensional space a triangle such as that shown in Fig. 2-4 would have two faces: a front face and a back face. Since faces do not need to be flat, this front-and-back combination of faces can actually enclose a lens-like cell. This configuration, having *two* faces, is called a *dihedron.* Since the faces both have three vertices, we call it a *trigonal* (or triangular) *dihedron.*

Figure 2-3 definitely requires modification in order to conform to our closure hypothesis, for both of its vertices are "dangling." Closure requires at least two edges terminating at each vertex. Hence Fig. 2-7 represents the simplest closed two-vertex structure we shall study. The admission of curved edges and faces makes this a very acceptable pattern: there are two faces (front and back), each having two vertices and two edges. It is therefore called a *digonal dihedron.* The dihedra are usually overlooked because only planar-faced polyhedra are considered, but in our more general structure theory they are important and should not be omitted in any exhaustive enumeration. We shall see later on that, when transformations of polyhedra are considered, these dihedra give rise to some very familiar polyhedra.

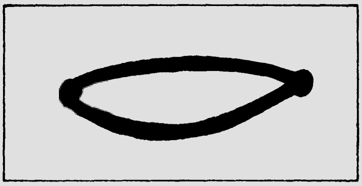


FIGURE 2-7 Digonal dihedron

I stated earlier that our structures are systems of interconnected elements of different dimensionality. A number of edges meet at each vertex: we shall call this number r, and call it the *valency* of the vertex toward edges. Analogously, a number of faces come together at a vertex: we shall call this number the valency of the vertex toward faces, and call it p.

Anyone who has observed a soap froth may have noticed that invariably four edges and six faces meet at a vertex: the vertex valencies are 4 toward edges, and 6 toward faces. When eight cubes are stacked so that all eight share a common vertex, that central vertex has an edge valency of 6 and a face valency of 12. Moreover, the fact that *eight* cubes meet at that vertex is expressed in terms of the *cell valency* (or valency toward cells) of that vertex, which would be 8. In general, the number of cells meeting at a vertex is called the *cell valency* (or valency toward cells) of that vertex, q.

In turn, every edge joins a number of vertices, faces, and cells. Every edge joins exactly *two* vertices: the *vertex valency* of an edge is always 2. On a two-dimensional surface every edge joins exactly two faces, but generally (e.g., in the soap froth or the stacked cubes) the *face valency* of edges is larger, and will be denoted by the symbol s. We have now the means of expressing the fact that a graph, pattern, or structure is two-dimensional in quantitative terms: every one of its edges has a face valency $s = 2$.

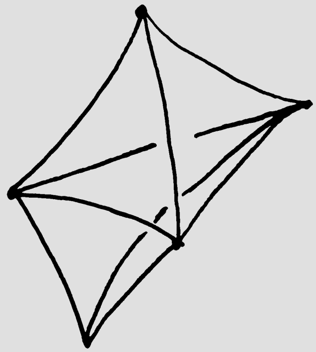
The number of cells meeting at an edge is exactly the same as the number of faces meeting at that edge, for every cell lies between two of the faces, and every face between two of the cells: they alternate. Hence the *cell valency* of edges is also s.

Analogously, a face has equal numbers of vertices and edges: we denote the *vertex valency* and *edge valency* of a face by n. Two cells meet at every face: the *cell valency* of all faces equals 2. In this connection it is well to remember that a closed surface divides space into two cells: the enclosed space and the outside space. The faces on a closed surface (e.g., a cube) in this sense have indeed a cell valency of 2.

Finally, each cell has a number of vertices, edges, and faces: we call the *vertex, edge,* and *face valencies* of a cell, respectively, m, l, and k. These valency definitions are summarized in Table 2-1.

TABLE 2-1
Valencies in Three-Dimensional Structures

Element	*Vertex valency*	*Edge valency*	*Face valency*	*Cell valency*
Vertex	–	r	p	q
Edge	2	–	s	s
Face	n	n	–	2
Cell	m	l	k	–



3

The Euler-Schlaefli Equation

In 1752 Leonhard Euler formulated a relationship between the numbers of elements of different dimensionality in a structure, which was generalized very elegantly by Schlaefli exactly one hundred years later. Schlaefli put his relationship in the following form:

$$\sum_{i=0}^{j} (-1)^i N_i = 1 + (-1)^j \tag{3-1}$$

In this concise symbolic expression N_i represents the number of elements of dimensionality *i:* N_0 would be the number of vertices, N_1 the number of edges, N_2 the number of faces, N_3 the number of cells. Schlaefli went on to corresponding four-, five-, etc., dimensional elements. The sign $\sum_{i=0}^{j}$ indicates that he sums these elements, starting with the zero-dimensional elements, going through to the j-dimensional elements for a j-dimensional structure. To understand the symbol $(-1)^i$, remember that when (-1) is squared it equals $+1$, that $(-1)^4$ also equals $+1$, but that $(-1)^3 = (-1)^5 = (-1)$. Generally, $(-1)^i = +1$ for even values of i, and $(-1)^i = -1$ for odd values of i. The right-hand side of equation (3-1) then equals *zero* for odd-dimensional structures, 2 for even-dimensional spaces.

We shall limit ourselves to dimensionalities not exceeding 3. To avoid subscripts and to use a more mnemonic code, we shall define the following symbols:

$$V \equiv N_0 = \text{number of vertices}$$
$$E \equiv N_1 = \text{number of edges}$$
$$F \equiv N_2 = \text{number of faces}$$
$$C \equiv N_3 = \text{number of cells}$$

For a three-dimensional structure $j = 3$ in equation (3-1), so that

$$V - E + F - C = 0 \tag{3-2}$$

We have seen that a two-dimensional structure that is singly connected and closed divides space into two cells, so that $C = 2$ for these structures; hence

$$V - E + F = 2 \tag{3-3}$$

This same expression would be obtained from equation (3-1) by setting $j = 2$. Such harmonious concordance is possible only if we do take the point of view that a closed, singly connected surface divides space into two cells (inside and outside the enclosure), and justifies the time and effort expended in the previous chapter on this point. A proof by mathematical induction of equation (3-2) consists of two parts:

1. Show that it holds for a particular value of V.
2. Show that, *if* it holds for a structure having V vertices, it will also hold for a structure having $(V + 1)$ vertices.

Figure 3-1 represents a tetrahedron: it has four vertices, six edges, and four faces, and divides space into two cells, having a two-dimensional surface. Hence $V - E + F - C = 4 - 6 + 4 - 2 = 0$, so that part 1 of the proof is completed.

Next let us suppose that we have a three-dimensional structure made up of tetrahedral cells only (Fig. 3-2). For our proof by mathematical induction we need to show that, *if* equation (3-2) holds, addition of one vertex will affect the numbers of vertices, edges, faces, and cells in such a way that equation (3-2) also holds for the new numbers.

Let us add on to our structure exemplified by Fig. 3-2 *one* tetrahedral cell that shares a face with a cell on the surface of the original

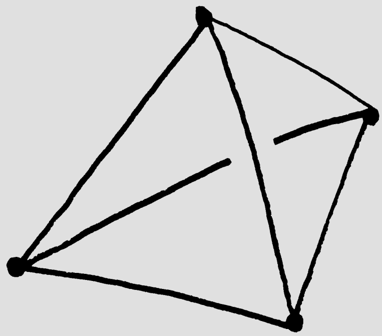


FIGURE 3-1 Tetrahedron: V=4, E=6, F=4, C=2

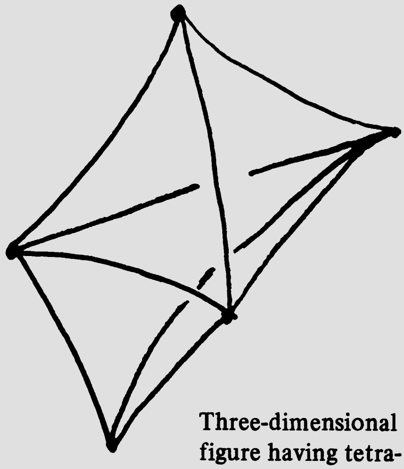


FIGURE 3-2 Three-dimensional figure having tetrahedral cells

structure. If we indicate the parameters of the new structure by primes:

$V' = V + 1$ because a single vertex was added on.

$E' = E + 3$ because this vertex must be joined to three of the original vertices.

$F' = F + 3$ because the new cell brings in three *new* faces, fusing its fourth one with the original structure.

$C' = C + 1$

Therefore

$$V' - E' + F' - C' = (V - E + F - C) + (1 - 3 + 3 - 1) = 0$$

Thus we have shown that *any* structure made up of tetrahedral cells obeys equation (3-2).

We shall next generalize to a single polyhedron that has triangular faces. Such a polyhedron might be inscribed on a spherical surface, since we have allowed our elements to be curved. We shall prove later on that the number of vertices chosen for such a polyhedron is not arbitrary, but is restricted to certain numbers. To determine this restriction, however, we need to ascertain that equation (3-2) is indeed valid for any polyhedron that has triangular faces.

To do so, we again add tetrahedral cells onto an original tetrahedron, but this time we destroy the face where they are joined. Instead of a structure made up of tetrahedral cells, we then create a singly connected polyhedron having a triangulated surface. The reasoning by mathematical induction is the same as before, but the parameters of a new structure are now

$V' = V + 1$	as before
$E' = E + 3$	as before
$F' = F + 2$	because three new faces *replace* the original one.
$C' = C = 2$	because throughout the growth of the structure it divides space into the two cells, inside and outside.

Since for the original structure $(V - E + F - C) = 0$, we find again for the new structure (the polyhedron): $V' - E' + F' - C' = 0$, q.e.d.

In particular, for a singly connected polyhedron $C = 2$, so that for such structures

$$V - E + F = 2 \tag{3-3}$$

Generalizing further, we shall abandon the requirement that the polyhedron be triangulated. Suppose that we had a polyhedron having all but one of its faces triangulated. That exceptional face has edge and vertex valency n. We can choose an additional vertex in or outside that face, and connect it by means of n new edges to the n vertices of the original face. The parameters change as follows as a result of this transformation:

$V' = V + 1$	
$E' = E + n$	
$F' = F + n - 1$	because the original face is *replaced* by n new ones.

$$C' = C = 2$$

$$\therefore V' - E' + F' = V - E + F$$

so that equation (3-3) is shown to be valid for both the old and the new structure. Since we have shown that triangulating a polygonal face of a polyhedron does not affect the expression $(V - E + F)$, any polyhedron can be transformed into a triangulated one without altering $(V - E + F)$, so that equation (3-3) is valid for any polyhedron.

Finally, we shall show that a structure having *any* collection of polyhedral cells obeys equation (3-2). To do so, we must realize that any polyhedron can be subdivided into tetrahedral cells by choosing an additional vertex inside the polyhedron, connecting it by m edges to the m vertices of the polyhedron. If, to begin with, we consider a structure that has all except *one* cell tetrahedral, that exceptional cell has respective vertex, edge, and face valencies equal to m, l, and k (cf. Table 2-1). Since this exceptional cell is a polyhedron, equation (3-3) applies to it:

$$m - l + k = 2 \tag{3-4}$$

Subidviding this exceptional cell into tetrahedral cells has the following effect on the parameters of the structure (we indicate a change in a parameter by the prefix Δ):

$\Delta V = 1$

$\Delta E = m$ because m edges join the new vertex to the original ones.

$\Delta F = l$ because every original edge of the subdivided cell is joined to the new vertex by a new face.

$\Delta C = k - 1$ because the original polyhedral cell is *replaced* by as many new cells as there were original faces.

This transformation therefore causes the following change in $(V - E + F - C)$:

$$\Delta(V - E + F - C) = 1 - m + l - k + 1$$

From equation (3-4) it follows that $\Delta(V - E + F - C) = 0$.

Since we have thus shown that subdividing polyhedral cells into tetrahedral cells does not affect $(V - E + F - C)$, and since equation (3-3) was proved for structures having tetrahedral cells, we have now proved this equation as well for structures having any polyhedral cells.* In addition, we impose an important restriction on three of the valencies listed in Table 2-1, in the form of equation (3-4).

An analogous restriction applies to vertex valencies. Let us consider a vertex having respective edge, face, and cell valencies, r, p, and q (cf. Table 2-1). Imagine a sphere around this vertex. The r edges emanating from the vertex will intersect with this sphere at r points, which we shall make the vertices of a polyhedron inscribed on the sphere. The p faces emanating from the vertex then generate p edges for the polyhedron inscribed on the sphere, and the q cells correspond to q faces on the sphere. Equation (3-3) applies to the polyhedron on the sphere:

$$r - p + q = 2 \tag{3-5}$$

Thus, of the eight parameters (valencies) listed in Table 2-1, only six are independent, the remaining two being determined by equations (3-4) and (3-5).

*Strictly speaking, the proof applies to *triangulated* polyhedral cells, since the single polyhedral cell surrounded by tetrahedral cells is necessarily triangulated. However, since we have already shown $(V - E + F)$ to be invariant to triangulation of a single polyhedron, an analogous argument applies to the assembly of polyhedral cells whose interfaces would, in this proof, still be triangulated.

4

Statistical Symmetry

We have seen that a spatial structure or graph may be described by the numbers of elements of different dimensionality that constitute it, and by the valencies of these elements toward each other. We have seen, furthermore, that the numbers of elements of different dimensionality are interrelated by the Euler-Schlaefli relation (equations 3-1 and 3-2), and that the valencies are restricted by two relations derived from the Euler-Schlaefli relation (equations 3-4 and 3-5).

In the program outlined in the Introduction we shall proceed from the more general structures to the more specialized ones. Initially, we shall deal with structures in which all elements have their own individual valencies, and later on we find the implications of letting the elements become equivalent to each other. Such equivalence is described by the *symmetry* properties of a structure: a structure in which no elements are equivalent to each other is called *unsymmetrical.*

Remarkably, we find in the present chapter that even in totally unsymmetrical structures the values of the valencies, when *averaged* over corresponding vertices, edges, or faces, are subject to some fundamental constraints. In symmetrical structures these *average* values of valencies become *the* valencies, and the constraints become symmetry relations. For this reason we sometimes refer to the statistical relations between valencies as *statistical symmetry.*

To relate the valencies to each other, we shall find relations between the *numbers* of vertices, edges, faces, and cells on the one hand, and their *valencies* toward each other on the other hand. The total number of edges in a structure can be found from the vertex valencies by summing the number of edges converging at every vertex over all vertices; it can also be found by summing the number of edges surrounding every face over all faces, or by summing the number of edges of each cell over all cells.

If we designate as V_r the number of vertices whose edge valency is r, then summing over all valencies of the vertices produces the expression

$$2V_2 + 3V_3 + 4V_4 + \ldots$$

because *two* edges terminate at *bivalent* vertices, of which there are V_2, *three* edges terminate at *trivalent* vertices, of which there are V_3, *four* edges terminate at each of the V_4 tetravalent vertices, etc. The summation is written in shorthand as follows:

$$2V_2 + 3V_3 + 4V_4 + \ldots = \sum_{r=2}^{\infty} rV_r$$

Since each edge terminates at two vertices, it is counted twice in this summation; hence

$$2E = \sum_{r=2}^{\infty} rV_r \tag{4-1}$$

Also, the total number of *vertices* equals the sum of all vertices having each valency:

$$V = \sum_{r=2}^{\infty} V_r \tag{4-2}$$

We can define the *average* edge valency of the vertices as follows:

$$\bar{r} = \frac{\sum_{r=2}^{\infty} rV_r}{V} \tag{4-3}$$

Hence (from equations 4-1 and 4-3)

$$\bar{r}V = 2E \tag{4-4}$$

If we count edges by summing over *faces*, the situation is somewhat more complicated, for generally s faces meet at an s-valent edge (cf. Table 2-1). The total number of *links* from faces to edges is

$$\sum_{n=2}^{\infty} nF_n$$

where F_n is the number of faces having edge valency n. Analogously, the number of links from edges to faces equals

$$\sum_{s=2}^{\infty} sE_s$$

where E_s is the number of edges having face valency s. The number of edge-to-face links must equal the number of face-to-edge links:

$$\sum_{s=2}^{\infty} sE_s = \sum_{n=2}^{\infty} nF_n \tag{4-5}$$

If we define average valencies as follows:

$$\bar{s} = \frac{\sum_{s=2}^{\infty} sE_s}{E} \quad \text{and} \quad \bar{n} = \frac{\sum_{n=2}^{\infty} nF_n}{F}$$

then

$$\bar{s}E = \bar{n}F \tag{4-6}$$

Analogously, if C_l equals the number of cells having edge valency l, then

$$\sum_{s=2}^{\infty} sE_s = \sum_{l=2}^{\infty} lC_l$$

(Remember that the face valency and cell valency of edges are equal to each other; both are denoted by s. Cf. Table 2-1.)

Defining

$$\bar{l} = \frac{\sum_{l=2}^{\infty} lC_l}{C}$$

we obtain

$$\bar{s}E = \bar{l}C \tag{4-7}$$

We can continue analogously with vertex–face linkages. Equating the number of vertex-to-face links to the number of face-to-vertex links, we obtain

$$\sum_{p=2}^{\infty} pV_p = \sum_{n=2}^{\infty} nF_n$$

Defining

$$\bar{p} = \frac{\sum_{p=2}^{\infty} pV_p}{V} \quad \text{and} \quad \bar{n} = \frac{\sum_{n=2}^{\infty} nF_n}{F}$$

we then obtain

$$\bar{p}V = \bar{n}F \tag{4-8}$$

Vertex–cell linkages give, analogously,

$$\sum_{q=2}^{\infty} qV_q = \sum_{q=2}^{\infty} mC_m$$

$$\bar{q} = \frac{\sum_{q=2}^{\infty} qV_q}{V}; \quad \bar{m} = \frac{\sum_{q=2}^{\infty} mC_m}{C}$$

and

$$\bar{q}V = \bar{m}C \tag{4-9}$$

Finally, the face–cell linkage gives

$$2F = \sum_{k=2}^{\infty} kC_k$$

for every face is a boundary between exactly two cells.

If we define

$$\bar{k} = \frac{\sum_{k=2}^{\infty} kC_k}{C}$$

then

$$2F = \bar{k}C \tag{4-10}$$

These equations are an implicit result of the assumption that there be no "dangling" vertices, edges, or faces, but that our structures be continuous and closed, having all valencies at least equal to 2. We can therefore call these equations our *continuity* equations. For convenience we summarize them as follows:

$$\bar{r}V = 2E \quad (4\text{-}4); \qquad \bar{s}E = \bar{n}F \quad (4\text{-}6); \qquad 2F = kc \quad (4\text{-}10)$$

$$\bar{p}V = \bar{n}F \quad (4\text{-}8); \qquad \bar{s}E = \bar{l}C \quad (4\text{-}7)$$

$$\bar{q}V = \bar{m}C \quad (4\text{-}9)$$

Here the first line relates elements whose dimensionalities differ by a single unit, the second relates elements whose dimensionalities differ by two units, the third relates elements whose dimensionalities differ by three units. There is a curious structure in this set of equations:

$\bar{s}E$ occurs in both (4-6) and (4-7)

$\bar{n}F$ occurs in both (4-6) and (4-8)

Equations (4-6), (4-7), and (4-8) are thus linked as follows:

$$\bar{p}V = \bar{s}E = \bar{n}F = \bar{l}C$$

From these, we express V, F, and C in terms of E:

$$V = (\bar{s}/\bar{p})E; \qquad F = (\bar{s}/\bar{n})E; \qquad C = (\bar{s}/\bar{l})E$$

Substitution of these expressions into equations (3-2) follow by division of both sides of the resulting equation by sE yields

$$\frac{1}{\bar{p}} - \frac{1}{\bar{s}} + \frac{1}{\bar{n}} - \frac{1}{\bar{l}} = 0 \qquad (4\text{-}11)$$

Equation (4-11) is very fundamental in structure theory; it interrelates average values of one of the valencies for each dimensionality. It followed from three of the six continuity equations. The remaining three continuity equations can be combined with the "linked set" as follows:

$$\begin{aligned} \bar{r}V &= 2E \ (4\text{-}4) \\ \bar{p}V &= \bar{s}E \end{aligned} \qquad \therefore \ \frac{1}{2}\bar{r} = \bar{p}/\bar{s} \qquad (4\text{-}12)$$

$$\begin{aligned} 2F &= \bar{k}C \,(4\text{-}10) \\ \bar{n}F &= \bar{l}C \end{aligned} \quad \therefore \frac{1}{2}\bar{k} = \bar{l}/\bar{n} \qquad (4\text{-}13)$$

$$\begin{aligned} \bar{q}V &= \bar{m}C \,(4\text{-}9) \\ \bar{p}V &= \bar{l}C \end{aligned} \quad \therefore \bar{p}/\bar{q} = \bar{l}/\bar{m} \qquad (4\text{-}14)$$

To these four statistical relationships between valencies should be added the following restrictions derived previously for *every* vertex and *every* cell:

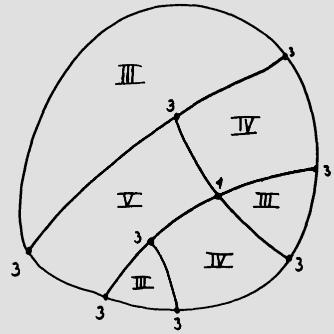
$$m - l + k = 2 \qquad (3\text{-}4)$$

$$r - p + q = 2 \qquad (3\text{-}5)$$

When the expressions for q and m from equations (3-5) and (3-4) are substituted into equation (4-14), and in turn r and k are eliminated by means of equations (4-12) and (4-13), equation (4-11) results. The latter equation, while an elegant and useful summarizer, is thus not independent of the others: there are *eight* valencies, and only *five independent* relationships. Thus *three* constraints need to be imposed to fix all valencies.

TABLE 4-1
Interrelationships Between Average Valancies

$\frac{1}{\bar{p}} - \frac{1}{\bar{s}} + \frac{1}{\bar{n}} - \frac{1}{\bar{l}} = 0$	(4-11)
$\frac{1}{2}\bar{r} = \bar{p}/\bar{s}$	(4-12)
$\frac{1}{2}\bar{k} = \bar{l}/\bar{n}$	(4-13)
$\bar{p}/\bar{q} = \bar{l}/\bar{m}$	(4-14)
$m - l + k = 2$	(3-4)
$r - p + q = 2$	(3-5)



5

Random Two-dimensional Nets

We shall first apply the equations derived in the previous chapter to two-dimensional nets. We have noted previously that two-dimensional structures are characterized by the fact that *all* edges have a face valency $s = 2$. According to equation (4-12) this implies for two-dimensional structures $\bar{r} = \bar{p}$; hence from equation (3-5), $\bar{q} = 2$. Equation (4-11) then becomes

$$\frac{1}{\bar{r}} + \frac{1}{\bar{n}} = \frac{1}{2} + \frac{1}{\bar{l}}$$

Two-dimensional structures divide space into two cells: $C = 2$. From equation (4-7), then, $l = E$, so that

$$\frac{1}{\bar{r}} + \frac{1}{\bar{n}} = \frac{1}{2} + \frac{1}{E} \tag{5-1}$$

There is a fundamental difference between the structures of equation (5-1) for two-dimensional structures and of equation (4-11) for three-dimensional structures. The former contains, in addition to valencies, the *total* number of edges, E; the latter is a relation between *valencies only*. In discussing two-dimensional structures, we shall therefore always need to specify the total number of edges. Instead of eight parameters related by five independent relationships, in three dimensions, we find in two dimensions *three* parameters—$\bar{r}$, $\bar{n}$, and E—and a single relationship between them.

In Fig. 5-1 we find an irregular structure having a variety of valencies. Such structures are frequently found on metallic surfaces, or on cracked ceramic surfaces (e.g., teacups!). We shall, as an illustration, apply equation (5-1) to the structure shown in Fig. 5-1. To do so properly, we remember that we postulated a closed structure—i.e., no "dangling" edges; we imagined the structure inscribed on a very large sphere. Therefore, in averaging over all faces, we must remember that the "outside" is a face, which in this case has *six* edges.

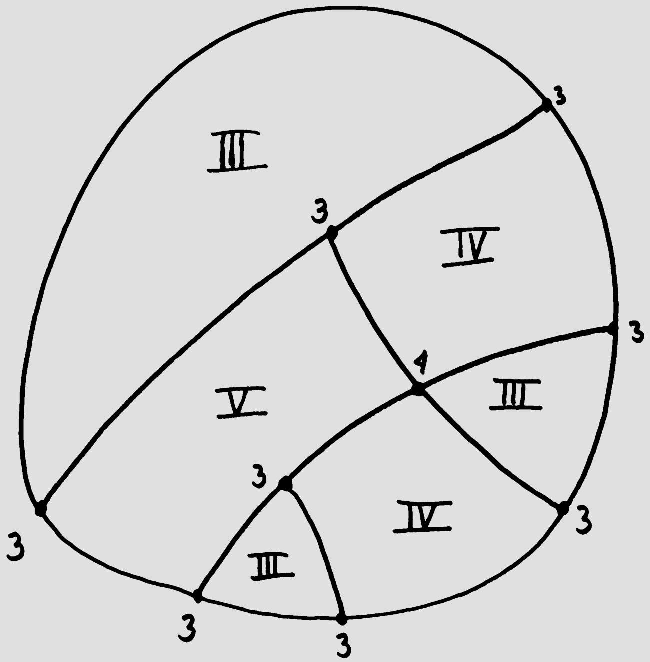


FIGURE 5-1 An irregular two-dimensional structure. Vertex valencies of faces are indicated by Roman numerals, edge valencies of vertices by Arabic numerals.

Hence

$$\bar{r} = \frac{8 \times 3 + 1 \times 4}{9} = \frac{28}{9};$$

$$\bar{n} = \frac{3 \times 3 + 2 \times 4 + 1 \times 5 + 1 \times 6}{7} = \frac{28}{7}$$

$$\therefore \frac{1}{\bar{r}} + \frac{1}{\bar{n}} = \frac{1}{2} + \frac{1}{14}$$

$$\therefore\ E = 14 \text{ (checks with figure)}$$

This computation is comparatively simple, because we have taken the "closed-system" point of view. Ordinarily this is not done; the "outside" six-edged face is not taken into account. The average value of n is then defined as follows:

$$\bar{n}' = \frac{\Sigma' n F_n}{\Sigma' F_n}$$

where the symbol Σ' indicates summation *excluding* the "outside." In terms of our definitions:

$$\Sigma' F_n = \sum_n F_n - 1 = F - 1$$

$$\Sigma' n F_n = \Sigma n F_n - E_c$$

where E_c equals the number of circumferential edges (in our example $E_c = 6$). Hence equation (4-6) becomes:

$$2E = \bar{n}F = \Sigma n F_n = \Sigma' n F_n + E_c$$

$$= \bar{n}'\Sigma' F_n + E_c = \bar{n}'(F - 1) + E_c$$

$$\therefore\ F = 1 + \frac{2E - E_c}{\bar{n}'}$$

Then equation (5-1) assumes the form

$$\frac{1}{\bar{r}} + \frac{1 - (E_c/2E)}{\bar{n}'} = \frac{1}{2} + \frac{1}{2E} \tag{5-1'}$$

When equations (5-1) and (5-1′) are compared, (5-1) is simpler and more elegant, hence more apt to be "fundamental." Yet they express the same concept, differing only in a definition of n and n'. From a practical point of view, however, the superiority of equation (5-1) is illustrated when the values of Fig. 5-1 are substituted into equation (5-1′):

$$\bar{n}' = \frac{3 \times 3 + 2 \times 4 + 1 \times 5}{6} = \frac{11}{3}$$

$$\frac{9}{28} + \frac{1 - (6/2E)}{11} \times 3 = \frac{1}{2} + \frac{1}{2E}$$

Eventually, of course, this expression also yields $E = 14$, but it is obvious that the arithmetic for this "open" system is much more laborious than that for our closed system. Since practical problems usually involve systems larger than that of Fig. 5-1, metallurgists are well advised to use equation (5-1) rather than (5-1′), remembering to take into account in the definition of n the "outside" face.

From equation (5-1) (and also from 5-1′) it is seen that for very large systems E becomes very large, and equations (5-1) and (5-1′) approach

$$\frac{1}{\bar{r}} + \frac{1}{\bar{n}} = \frac{1}{2} \tag{5-2}$$

Even for the example of Fig. 5-1 the term dependent on E constitutes only 12.5% of the right-hand side of equation (5-1). It is therefore understandable that the simplicity of equation (5-2) would appeal to those working with large irregular systems, in which the value of E is large, but also rather irrelevant.

Cyril S. Smith[1] has stated that under certain conditions it is even possible to use equation (5-2) for a *bounded* net—namely, if proper fractional corrections are made for circumferential effects. No formal procedure is given for making these corrections, but for the particular examples given, common sense appears to be a good guide, and equation (5-2) works out empirically.

We should be able to surmise what Cyril Stanley Smith's "common-sense" corrections amount to formally by starting with equation (5-1), and defining:

$$\frac{1}{\bar{\rho}} = \frac{1}{\bar{r}} - \frac{1}{E}$$

From equation (4-4) by substitution:

$$\frac{1}{\bar{\rho}} = \frac{1}{\bar{r}}\,\frac{V-2}{V}$$

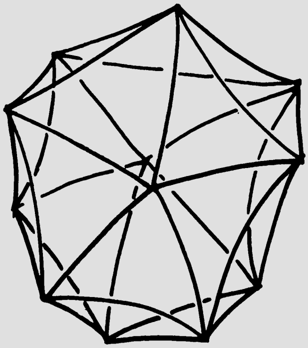
[1] In *Hierarchical Systems,* L. L. Whyte, A. G. Wilson, and D. Wilson, eds. (American Elsevier, New York, 1969), p. 71.

$$\bar{\rho} = \frac{\bar{r}V}{V-2} = \frac{\sum_r rV_r}{V-2} \tag{5-3}$$

Comparing equations (4-3) and (5-3), we discover that equation (5-2) will work as well as (5-1) if, in the averaging procedure to find r, we simply deduct 2 from the total number of vertices. It would appear, then, that in making his common-sense fractional corrections for circumferential elements, Cyril Smith would have dropped in effect exactly two vertices.

Although we are only *surmising,* and could not *prove,* how Smith made his corrections, one explanation appears very plausible. If we straighten out all curved edges, the process is easier to follow; no radical alteration is introduced in doing so. An internal vertex in a plane is surrounded by r angles, which add up to 360°. If the vertex does not lie in a plane, but on a sphere, or at the intersection of r non-coplanar faces, the sum of the angles would be less than 360°. If V vertices lie in a plane, then the sum of all angles around all V vertices would be $V \times 360°$. However, for a closed structure either all V vertices are coplanar, but the structure folds back along a circumference to form a "back face," or the V vertices lie on a curved closed surface (e.g., a sphere or a polyhedron). The former case is the one considered by Cyril Smith, and exemplified in Fig. 5-1; for *either* case it has been shown[2] that the sum of *all* angles around *all* vertices falls exactly short of $V \times 360°$ by 720°; i.e., this sum is actually $(V-2) \times 360°$. It is not surprising, therefore, that C. S. Smith would, in allowing for the fractional loss of vertices along the circumference, have lost in toto *exactly two* vertices.

[2] Cf. R. Buckminster Fuller, with E. J. Applewhite and Arthur L. Loeb: *Synergetics* (Macmillan New York, 1975), pp. 342-343, 826.



6 Degrees of Freedom

In Chapter 1 we discussed the dimensionality of vertices, edges, faces, and cells. A vertex has dimensionality *zero;* on surfaces (dimensionality *two*) a vertex can move with *two* degrees of freedom, whereas on a curve (dimensionality *one*) it can move with only a single degree of freedom. In three-dimensional space a vertex has three degrees of freedom: three quantities are needed to specify its location.

A system of two vertices in a three-dimensional space needs twice three quantities to specify its configuration. These six quantities could be the separate three per vertex. However, one might, for example, choose a point at the center of a straight edge joining the two vertices. This point would require three coordinates to specify its location. The distance from this central point to the two vertices would be a fourth parameter necessary to specify the configuration, and two angles would specify the orientation of the straight edge—e.g., the elevation out of the horizontal plane, and the direction of the projection of the edge on the horizontal plane. In either description *six* quantities are required to specify the configuration: it has six degrees of freedom. If, however, we are interested only in the relation of the vertices with respect to each other, then only their distances from each other would be relevant, leaving the other five parameters irrelevant. The second description would be preferable for such a case.

A structure having three vertices would have *nine* degrees of freedom. Three quantities could specify the location of a point coplanar with and equidistant from the three vertices. Two more parameters would suffice to indicate the orientation of the plane in space—e.g., by specifying the direction of a line perpendicular to the plane that contains the three vertices. Three parameters would give the distances between the vertices, and the ninth parameter would indicate the orientation of the triangle so specified around the line perpendicular to the plane of the vertices. Of these nine parameters only the three distances between the vertices are needed to relate the three vertices to each other, the remaining six relating the configuration to its surroundings. The three distances are called *internal* degrees of freedom, the remaining six *external* degrees of freedom.

Four vertices have twelve degrees of freedom in three-dimensional space. Of these, three specify a point equidistant from these four vertices. If an axis is chosen through that equidistant point and one of the vertices, the orientation of this axis requires two angles, and the orientation of the entire configuration around this axis one additional angle. These six are the *external* degrees of freedom. The remaining six, *internal,* degrees of freedom are the six distances between the four vertices.

In general, a structure having V vertices will have $3V$ degrees of freedom, of which *six* are external—e.g., the location of a center, the orientation of an axis, and the oiientation around that axis. The remaining $(3V - 6)$ degrees of freedom are internal; they give us the *maximum* number of relations (edges) that may be *independently* specified between the V vertices.

A system of V vertices may have many more than $(3V - 6)$ possible interrelationships. *Each* vertex may be connected to $(V - 1)$ other vertices; this would result in $V(V - 1)$ connections emanating from all vertices. However, each connection emanating from a vertex also terminates at a vertex, so that we have in effect $\frac{1}{2}V(V - 1)$ *possible* connections, of which only $(3V - 6)$ may be independently assigned. When $(3V - 6) < \frac{1}{2}V(V - 1)$, the system has more possible connections than could be independently specified; when $(3V - 6) = \frac{1}{2}V(V - 1)$, all connections can be independently specified, and when $(3V - 6) > \frac{1}{2}V(V - 1)$, the system is not specified even when all its connections are.

We might exemplify these statements more concretely by saying that a structure having V vertices will be rigid only if $(3V - 6) \leqslant \frac{1}{2}V(V - 1)$ and all vertices are joined by rigid rods. However, only if $(3V - 6) = \frac{1}{2}V(V - 1)$ will it be possible to choose the length of *all*

$\frac{1}{2}V(V-1)$ independently: for no structure is it possible to choose *more* than $(3V-6)$ of the $\frac{1}{2}V(V-1)$ rod lengths independently.

Let us solve the critical equation $(3V-6) = \frac{1}{2}V(V-1)$:

$$\frac{1}{2}\,V(7-V) = 6$$

Since $\frac{1}{2}V > 0$ and $6 > 0$, we require $(7-V) \geqslant 0$, hence $V \leqslant 7$. Inspection shows that only two value of V satisfy this condition: $V = 3$ and $V = 4$.

In three-dimensional space there are, therefore, only two rigid structures in which *all* distances may be independently specified (in particular all equal to each other)—namely, the *triangle* and the *tetrahedron*. For instance, when $V = 5$, one might arrange the vertices in a triangular bipyramid (Fig. 6-1) or a square pyramid (Fig. 6-2), but in no case could all *ten* connections be made equal in length: for the bipyramid nine edges could at most be equal to each other, with the two trivalent vertices at different distances from each other, whereas in the square pyramid vertices along the square base would be at two different distances from each other (along an edge and diagonally across).

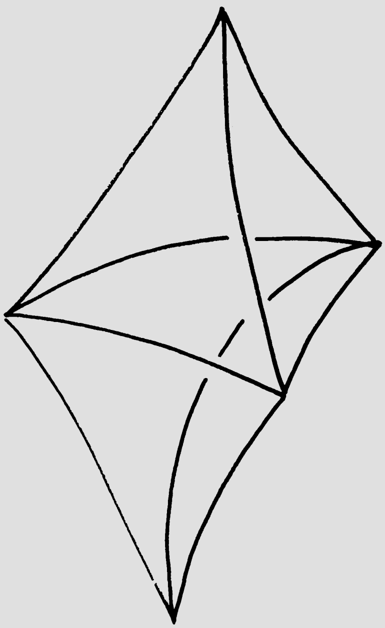


FIGURE 6-1 Triangular bipyramid

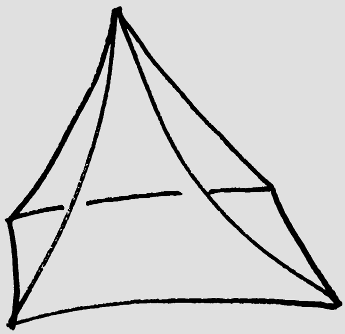


FIGURE 6-2 Square pyramid

The triangular bipyramid has nine edges and nine degrees of freedom: when all nine edges are specified (rigid rods), the structure is specified (rigid). The square pyramid has only eight edges, but also has nine degrees of freedom, hence will not be rigid unless one of the basal diagonals is also specified (rigid).

It will be important to discover under which conditions a polyhedron would be rigid. From equation (4-4) we know that $\bar{r}V = 2E$. We also know that all degrees of freedom are removed when $E = 3V - 6$ Hence a necessary condition is

$$\left.\begin{aligned} \bar{r}V &= 6V - 12 \\ \therefore \frac{1}{\bar{r}} &= \frac{V}{6(V-2)} \\ \bar{r} &= 6 - \frac{12}{V} \end{aligned}\right\} \qquad (6\text{-}1)$$

Moreover, from equation (5-1), with $E = 3V - 6$:

$$\frac{1}{\bar{n}} = \frac{1}{3}; \qquad \therefore \bar{n} = 3$$

Since a polyhedron would not have digonal faces ($n = 2$), this means that $n = 3$. Thus we have proved that triangulation is necessary for making a polyhedron rigid. This type of analysis provides a means for designing stable configurations: a structure will not be rigid unless, of its $\frac{1}{2}N(N-1)$ possible connections, $(3N-6)$ are rigid.

For $V = 5$ we considered the triangular bipyramid, and the square pyramid and noted that the latter needs at least a rigid diagonal to stabilize its base. Both structures then have only triangular faces. The valencies of the vertices of the triangular bipyramid are as follows:

$r = 3$ for both polar vertices

$r = 4$ for the three equatorial vertices

For the stabilized pyramid:

$r = 4$ for the "top" or apex of the pyramid

$r = 4$ for the two base vertices joined by the diagonal

$r = 3$ for the remaining two vertices.

Both structures thus have *six* triangular faces, two trivalent and three quadrivalent vertices. In both, the two trivalent vertices are joined only to quadrivalent ones. Thus, apart from actual lengths of connections, the two are entirely equivalent. Their average valencies are $\bar{n} = n = 3, \bar{r} = (18/5)$, in agreement with equation (6-1).

For $V = 6$, equation (6-1) gives $r = 4$; the simplest triangulated polyhedron here is the octahedron, having all vertex valencies $r = 4$.

For $V = 7$, equation (6-1) yields $\bar{r} = (30/7)$. The simplest figure corresponding to these numbers is a pentagonal bipyramid. The two polar vertices are pentavalent; along the equator are five vertices having valency 4, each being joined to two other equatorial vertices and to the two polar ones (Fig. 6-3).

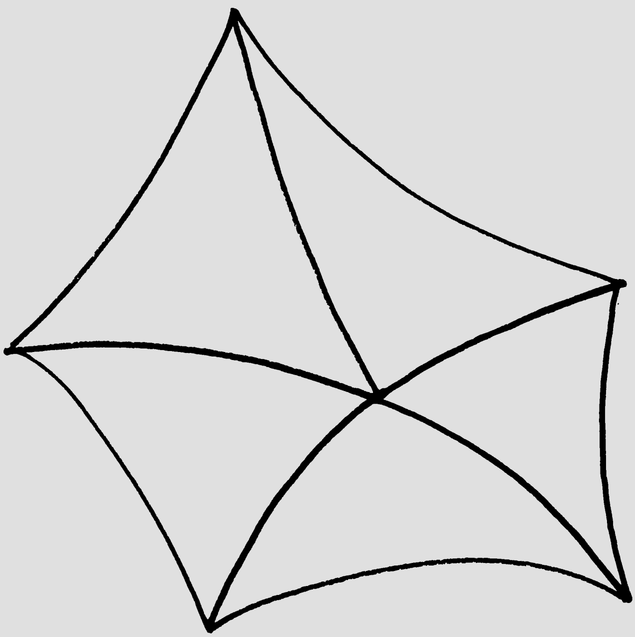


FIGURE 6-3 Top half of pentagonal bipyramid, showing one polar, and five equatorial vertices

We can readily see that any bipyramid obeys equation (6-1). If the equator is a j-gon, $V = (j + 2)$. The two polar vertices are j-valent, while the j equatorial vertices are 4-valent:

$$\bar{r} = \frac{j \times 4 + 2 \times j}{j + 2} = \frac{6j}{j + 2}$$

From equation (6-1):

$$\frac{6j}{j + 2} = 6 - \frac{12}{V}; \qquad \therefore V = j + 2, \text{q.e.d.}$$

For $V = 8$, $\bar{r} = 4\frac{1}{2}$. There are many different structures obeying this condition. There is the hexagonal bipyramid. There are also structures that can be considered hexahedra (e.g., a cube) whose faces are reinforced (triangulated) by means of a diagonal connection. The structure exemplified in Fig. 6-4 has two 3-valent and six 5-valent vertices

$$\bar{r} = \frac{2 \times 3 + 6 \times 5}{8} = 4\frac{1}{2}$$

In Fig. 6-5 there are four each of 3-valent and 6-valent vertices: in either case $\bar{r} = 4\frac{1}{2}$.

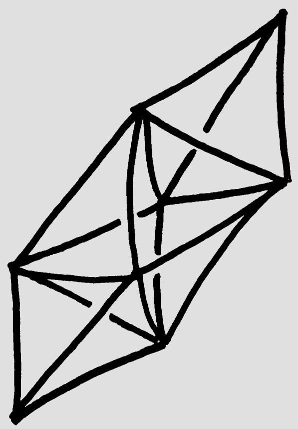


FIGURE 6-4 Eight vertices, six 5-fold, two 3-fold, $\bar{r} = 4\text{-}1/2$

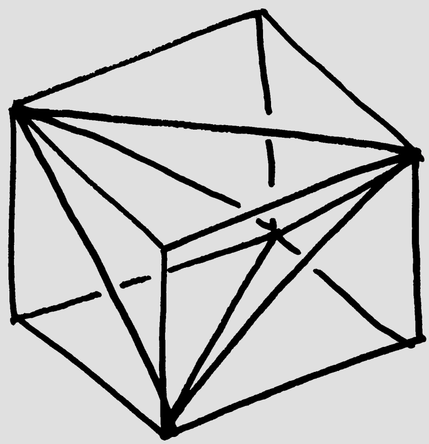


FIGURE 6-5 Eight vertices, four 6-fold, four 3-fold, $\bar{r} = 4\text{-}1/2$

It is clear that there are numerous possible triangulated structures. What is important is, however, that, regardless of *how* they are interconnected, V vertices require exactly $(3V - 6)$ interconnections to be stabilized. Stable structures are not necessarily triangulated polyhedra, for there are numerous internal struts that will stabilize a structure without making it a triangulated polyhedron. However, triangulated polyhedra have now been proved to be stable.

Because we have already observed that for any value of V there is a stable bipyramid, and because we have also found a multiplicity of different structures having any given combination of values of V, E, and $\bar{r}$, we shall skip over the structures having $V = 9$, 10, and 11: the principles are all stated, and nothing of additional interest appears to be found in these structures. With $V = 12$ we arrive at a very interesting set of structures, however. From equation (6-1), $\bar{r} = 5$ when $V = 12$. The structure having $\bar{r} = r = 5$, $V = 12$, is the *icosahedron,* which has $F = 20$, $E = 30$ (Fig. 6-6). It is also seen from equation (6-1) that the maximum value of $\bar{r}$ is 6, which can occur only when V is infinite. Hence the icosahedron is the largest structure of finite extent in which all faces are triangulated and all vertices have the same valency. If we wish to build a triangulated polyhedron having more than twelve vertices, we must have $5 < \bar{r} \leqslant 6$, with $\bar{r} = 6$ resulting in an infinitely extended structure. Accordingly, one would use a combination of 5-valent and 6-valent vertices; this is just what R. Buckminster Fuller has done in designing his geodesic domes. We

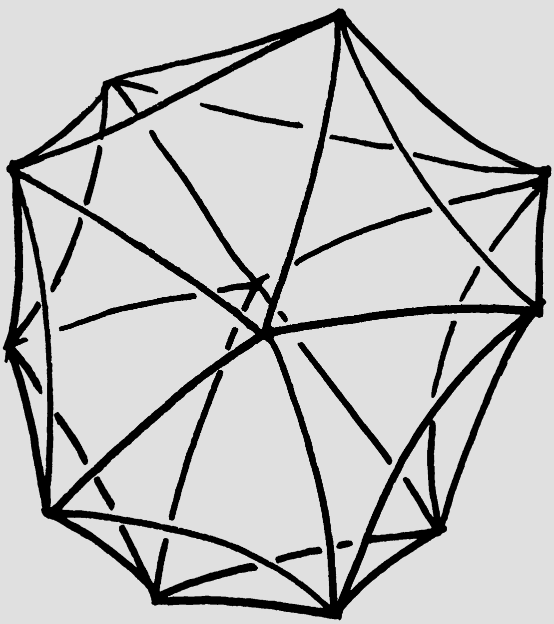


FIGURE 6-6 Icosahedron

can find exactly how many 5-valent vertices are needed for a complete polyhedron (the dome, of course, uses only part of the polyhedron). From equation (6-1):

$$\bar{r} = 6 - \frac{12}{V_5 + V_6}$$

where V_5 and V_6 are, respectively, the numbers of 5- and 6-valent vertices. Also

$$\bar{r} = \frac{5V_5 + 6V_6}{V_5 + V_6} \quad \text{by definition}$$

Equating these two expressions for $\bar{r}$, and multiplying both sides of the equation by $(V_5 + V_6)$, we find

$$6(V_5 + V_6) - 12 = 5V_5 + 6V_6$$
$$\therefore V_5 = 12$$

We conclude that, regardless of the number of 6-valent vertices, there must be exactly *twelve* 5-valent vertices.

This phenomenon—that in a polyhedron having two different vertex valencies, one of the valencies *must* be present in a fixed number—is sufficiently remarkable that it warrants a more general investigation. Let us consider, instead of triangular faces, more general n-gonal faces: the polyhedron has all faces equivalent. There are V_a vertices having valency r_a, and V_b vertices having valency r_b. Setting $s = 2$ in equation (4-6), we find, from equations (4-4) and (4-6),

$$r_a V_a + r_b V_b = nF \tag{6-2}$$

Moreover, since $V = E + 2 - F$, equation (4-6) yields

$$V_a + V_b = 2 + \left(\frac{1}{2}n - 1\right)F \tag{6-3}$$

Solving equations (6-2) and (6-3) simultaneously, we obtain

$$V_a = \frac{2r_b + \left[\left(\frac{1}{2}n - 1\right)r_b - n\right]F}{r_b - r_a} \tag{6-4}$$

$$V_b = \frac{2r_a + \left[\left(\frac{1}{2}n - 1\right)r_a - n\right]F}{r_a - r_b} \tag{6-5}$$

For instance, for the geodesic dome example just considered, $r_a = 5, r_b = 6$, and $n = 3$.

$$V_5 = \frac{12 + (3 - 3)F}{1}; \qquad V_6 = \frac{10 + \left(-\frac{1}{2}\right)F}{-1} \tag{6-6}$$

It is seen, therefore, that V_5 equals 12, regardless of the value of F, in accordance with our previous finding. Additionally, since V_b has to be nonnegative, $F \geqslant 20$, the equal sign referring to the icosahedron, for which $V_b = 0$.

We can generalize, choosing $r_b > r_a$, from equations (6-4) and (6-5), that V_a will be independent of F, and a function only of r_a and r_b if

$$r_b = \frac{2n}{n - 2} \tag{6-7}$$

in which case

$$V_a = \frac{2r_b}{r_b - r_a} \tag{6-8}$$

In this instance, also, $r_a < r_b < \frac{2n}{n-2}$, so that

$$F \geqslant \frac{-4r_a}{(n - 2)r_a - 2n} = \frac{4r_a}{(n - 2)(r_b - r_a)} \tag{6-9}$$

Conversely, V_b is independent of F if

$$r_a = \frac{2n}{n - 2}$$

in which case

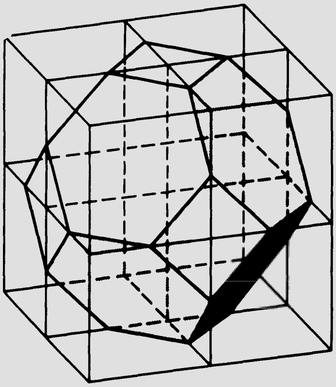
$$V_b = \frac{2r_a}{r_a - r_b} < 0$$

Since $V_b < 0$ is absurd, it is not possible to make V_b independent of F. However, for every value of n there is a value of r_b, given by equation (6-7), for which V_a is fixed at a value given by equation (6-8), and for which inequality (6-9) provides a minimal value of F.

As we shall, further on, exhaustively enumerate all polyhedra in which *either* all vertices *or* all faces are equivalent, or both, this is not the place to enumerate all possible combinations of n and r_b that would meet conditions (6-7), (6-8), and (6-9). However, for those particularly interested in the phenomenon of exactly *twelve* 5-valent vertices, and its generalization, but unwilling to consider the broader topic of the exhaustive enumeration, those relations will prove useful.

For instance, for $n = 4$, $r_b = 4$; this combination leaves only a possible $r_a = 3$, hence $V_a = 8$, and $F \geqslant 6$. The simplest structure having this restriction is the cube or, more generally, the hexahedron, but also interesting is the structure having $F = 12$, $V_b = 6$, the rhombohedral dodecahedron, which, of course, has also eight 3-valent vertices as well as the six 4-valent ones. What links the cube and the rhombohedral dodecahedron is our knowledge that a polyhedron having quadrilateral faces necessarily has exactly *eight* 3-valent vertices, regardless of how many 4-valent vertices (none for the cube, six for the dodecahedron) there are.

Summarizing, then, we find that a structure that has V vertices and E edges, has $(3V - E - 6)$ degrees of freedom; when $E < 3V - 6$ the structure cannot possibly be rigid; when $E = 3V - 6$ it can be rigid if the edges are properly applied. The maximum value of E equals $\frac{1}{2}V(V - 1)$; when $E > 3V - 6$, only $(3V - 6)$ of the edges can have their lengths independently specified.



7

Duality

We have taken a special point of view of structure, namely as a system of interlinked elements of different dimensionalities. The valencies describe these linkages. In general, in three dimensions, eight valencies are variable, and two (the vertex valency of every edge, and the cell valency of every face) are necessarily equal to 2.

There are certain relationships between diverse patterns or structures whose understanding enables us to transform one into the other. That is to say that, in addition to relationships *within* structures, we can consider relationships *among* structures, and the resulting transformations of structures. We have thus created a hierarchy of structure, whose description we might call the *structure of structures.*

The principal transformations to be considered here are: taking a dual, truncation, and stellation. We shall show that each of these involves the replacement of all or part of the elements of a given dimensionality by elements of a different dimensionality. Such replacement may occur only if the expression $(V - E + F - C)$ remains unchanged, and the linkages are conserved. In two-dimensional structures, the face valency of edges equals the vertex valency of the edges–i.e., 2. In such structures it is very convenient to interchange vertices and faces, for every face–face connection through an edge then becomes a vertex–vertex connection through an edge, and the sum $(V + F)$ in the expression $(V - E + F - C)$ remains unaltered. Two structures that are related by the fact that to every face in one there corresponds a vertex in the other, *and vice versa,* are called each other's *duals.*

In *three* dimensions the valency of faces toward cells equals 2, as does the valency of edges toward vertices. Here a duality relation could be defined between structures in which *every face and every cell* in one corresponds to an *edge-and-vertex* combination in the other. A transformation of faces/cells into edges/vertices would change the *sign* of the expression $(V - E + F - C)$, but since this expression is actually equal to *zero,* it remains in fact unchanged. Such generalized multidimensional dualities are being investigated by John Robinson.

Since in two dimensions $V - E + F = 2$, and duals have the same value of the sum $(V + F)$, duals necessarily have the same numbers of edges. Examples of dual pairs of polyhedra are listed in Table 7-1, and illustrated in Figs. 7-1 through 7-9.

TABLE 7-1
Examples of Dual Polyhedra

Polyhedron Pair	*V*	*F*	*E*	*Illustrated in Fig.*
Cube	8	6	12	7-1
Octahedron	6	8		7-2
Truncated octahedron	24	14	36	7-3
Stellated cube	14	24		7-4
Tetrahedron / Tetrahedron	4	4	6	7-5
Truncated tetrahedron	12	8	18	7-6
Stellated tetrahedron	8	12		7-7
Cuboctahedron	12	14	24	7-8
Rhombohedral dodecahedron	14	12		7-9

Two remarkable observations can be drawn from Table 7-1. In the first place, the tetrahedron has the same number of vertices and faces, hence is *its own* dual. It is therefore listed as *a pair.* The second observation is that, if two polyhedra are each other's duals, then a *stellation* of one is dual to a *truncation* of the other. For the

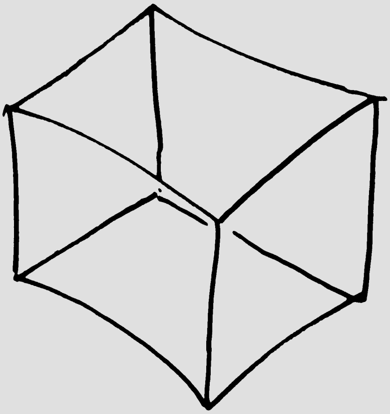


FIGURE 7-1 Cube

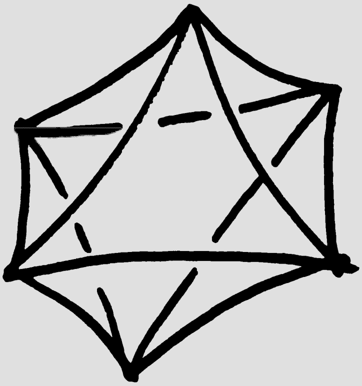


FIGURE 7-2 Octahedron

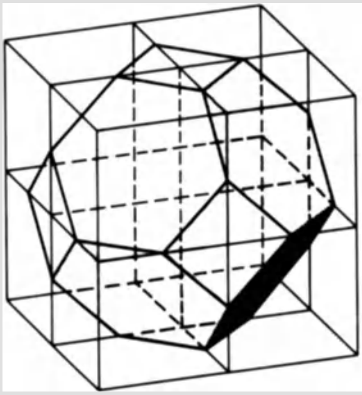


FIGURE 7-3 Truncated Octahedron shown inside a cube as frame of reference

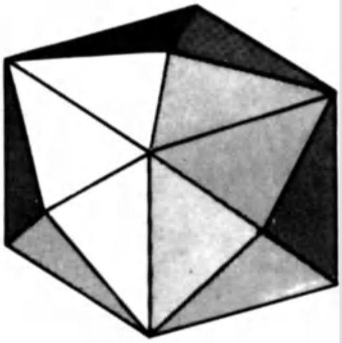


FIGURE 7-4 Stellated Cube

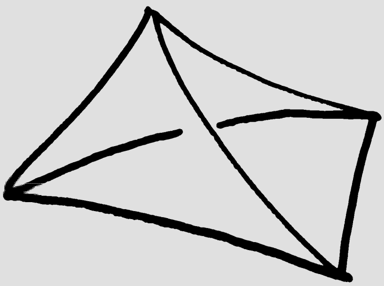


FIGURE 7-5 Tetrahedron

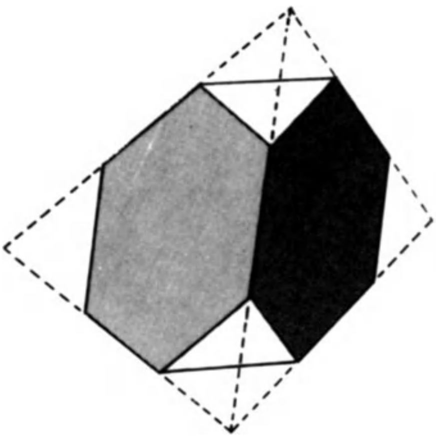


FIGURE 7-6 Truncated Tetrahedron

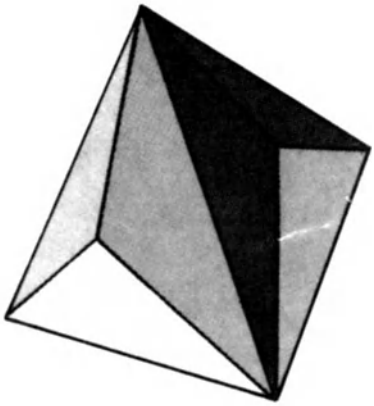


FIGURE 7-7 Stellated Tetrahedron

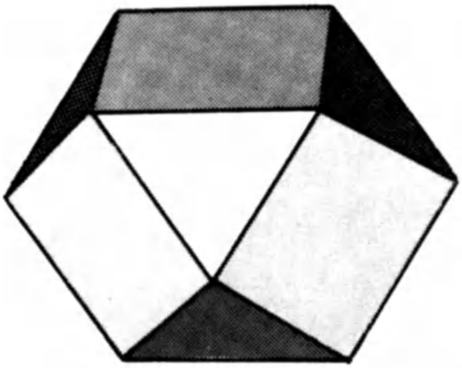


FIGURE 7-8 Cuboctahedron

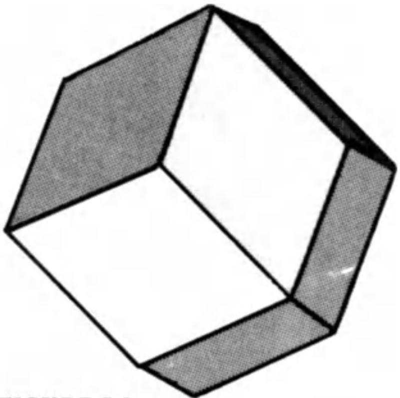


FIGURE 7-9 Rhombohedral Dodecahedron

time being, we shall leave this observation as an experimental phenomenon, and return to it when we study the stellation/truncation transformations more generally.

There is, accordingly, a *symmetry* in our structure of structures: to every structure that is not its own dual there corresponds a dual structure distinct from the first one. Any theorem that applies to a given structure can be transformed into a theorem applying to its dual. For instance, equations (6-4) and (6-5) give expressions for the numbers of vertices of respective edge valencies r_a and r_b in a polyhedron whose faces are all n-gons. These equations can be transformed into equations applying to polyhedra all of whose vertices have valency r, but whose faces are n_a-gons and n_b-gons:

$$F_a = \frac{2n_b + \left[\left(\frac{1}{2}r - 1\right)n_b - r\right]V}{n_b - n_a} \tag{7-1}$$

$$F_b = \frac{2n_a + \left[\left(\frac{1}{2}r - 1\right)n_a - r\right]V}{n_a - n_b} \tag{7-2}$$

By an argument analogous to the one used in the previous chapter we conclude that F_a is independent of V if

$$r = n_b\left(\frac{1}{2}r - 1\right); \qquad \text{i.e., } n_b = \frac{2r}{r - 2} \tag{7-3}$$

For example, if $r = 3$, $n_b = 6$, and

$$F_a = \frac{12}{6 - n_a} \tag{7-4}$$

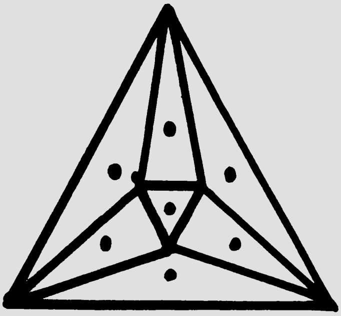
This means that for a polyhedron having $r = 3$ for all vertices an arbitrary number of hexagons combines with exactly *twelve pentagons* ($F_a = 12$ when $n_a = 5$), or with exactly *six quadrilaterals* ($F_a = 6$ when $n_a = 4$), or with exactly *four triangular* faces ($F_a = 4$ when $n_a = 3$). An example of the twelve pentagons combining with hexagons is a soccer ball: every one of these can be seen to have exactly a dozen black pentagonal patches as well as (usually but not necessarily) twenty white hexagonal ones. The six quadrilateral faces combining with hexagons turn up in the truncated octahedron (Fig. 7-3), the four triangular ones in the truncated tetrahedron (Fig. 7-6).

If $r = 4$, then F_a is independent of V if (cf. equation 7-3) $n_b = 4$; then (cf. equation 7-1):

$$F_a = \frac{8}{4 - n_a}$$

Hence we can have any number of quadrilaterals combining with exactly *eight* triangles; an example is the cuboctahedron (Fig. 7-8).

We shall return to equations (6-4), (6-5), (7-1), and (7-2) when we explore more exhaustively the possible combinations of valencies in two-dimensional (infinite as well as finite, polyhedral) structures.



8

Schlegel Diagrams

We have, throughout the previous chapters, repeated *ad nauseam* two principles: (1) Polyhedral surfaces are *two*-dimensional, and (2) we are not committed to definite distances and angles, but only to *numbers* of elements, their dimensionalities, and their valencies. Given these two principles one concludes that every polyhedron may be distorted such that it can be laid out flat on a surface so that no edges cross. This is done by choosing one particular face, extending it such that it becomes the frame within which the remainder of faces, edges, and vertices are contained. Visually this amounts to holding one face quite close to the eyes, looking at the structure through that face, and drawing the projection of the structure as seen in this exaggerated *perspective.* (Note that *per-spective* actually means "as seen through"!) Such a perspective projection of a polyhedron is called a Schlegel diagram. For our discussion Schlegel diagrams are important because they do not just *represent* our structures: they *are* our structures. Although we find it convenient to compare a structure having eight 3-valent vertices, six 4-valent faces, and twelve edges to the cube, so familiar to us, it can be equally well represented by the Schlegel diagram of Fig. 8-1, which has exactly the same elements and valencies as does the cube. The advantage of the Schlegel diagram is that all its elements and connections are explicitly visible, with no hidden elements. The fact that the face represented by the circumference of the Schlegel diagram must be counted explicitly as a face certainly need not be belabored in the context of this treatise (cf. Chapter 5).

In Figs. 8-1 ff. we show Schlegel diagrams equivalent to some of the polyhedra discussed in previous chapters. Because of the very fine illustrations of polyhedra in such books as Wenninger's,[1] Williams's,[2] and Critchlow's[3] and the above-mentioned visibility of all elements in the Schlegel diagrams, we shall principally make use of the latter illustrations rather than attempt to duplicate the three-dimensional views in the books referred to.

In the previous chapter we discussed the notion of *duality*. It would be nice at times to compare Schlegel diagrams of pairs of dual structures; Figs. 8-1 and 8-2 do *not* make the duality of the cube and the octahedron obvious. The problem is that the *face* that frames the entire Schlegel diagram will, in the transformation to the dual, produce a *vertex* that either lies *behind* the dual structure, or needs to represent the infinity of the unbounded surface in the original structure, depending on whether the original structure was interpreted as a (collapsed) polyhedron or as a finite tessellation on an infinite surface.

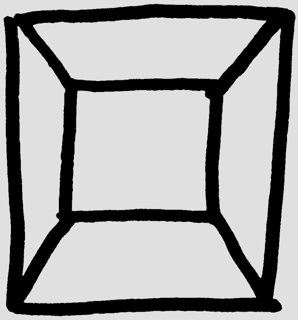


FIGURE 8-1 Schlegel diagram of a cube $r = 3, n = 4$

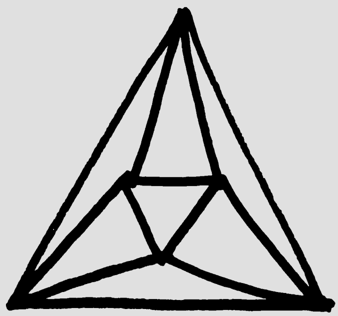


FIGURE 8-2 Schlegel diagram of an octahedron $r=4, n=3$

This problem can be interpreted slightly differently by comparing the Schlegel diagram to a polar map of the globe. Imagine a

[1] Magnus J. Wenninger: *Polyhedron Models* (Cambridge University Press, 1971).

[2] Robert Williams: *Natural Structure* (Endaemon Press, Mooroark, California, 1972).

[3] Keith Critchlow: *Order in Space: A Design Source Book* (Viking Press, New York, 1970).

plane tangent to the earth at the North Pole. By the use of distortion the whole surface of the earth may then be laid out on this tangent plane: the North Pole would be at the center, with the meridians radiating out, and the parallel circles centered on the North Pole. Regions of the earth farthest from the North Pole would be stretched enormously, but the set of radial lines and circles would amount to a Schlegel diagram of the sphere tessellation delineated by meridians and parallels. However, it would be quite impossible on this polar map to indicate the *South Pole:* it would be at infinity. We can, however, indicate its existence by putting an arrow at each of the meridians, indicating that *all* meridians terminate at a single vertex, the South Pole.

We propose here a similar device to define a *dual Schlegel diagram,* in which the vertex that in the dual structure corresponds to the "framing" face in the original is indicated by an arrow on each edge converging on that particular vertex. As examples, we derive from Figs. 8-1, 8-2, and 8-3 the *dual Schlegel diagrams* for the octahedron, cube, and tetrahedron in Figs. 8-6, 8-7, and 8-8.

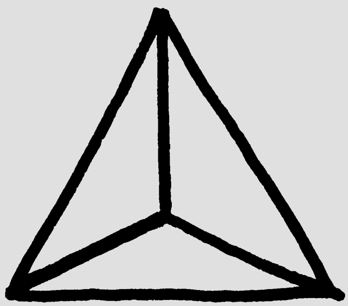


FIGURE 8-3 Schlegel diagram of a tetrahedron r=3, n=3

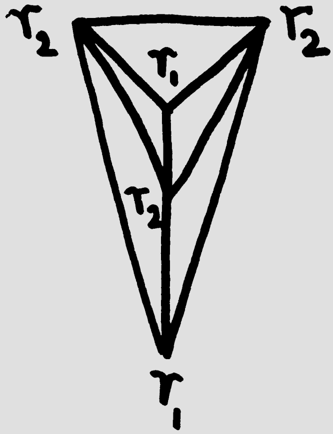


FIGURE 8-4 Triangular bipyramid r_a=3, r_b=4, n=3

(For an application of Schlegel-like mappings of regular and semiregular solids, cf. Athelstan Spilhaus: "Geo-Art: Tectonics and Platonic Solids," *Transactions of the American Geophysical Union,* **56** No. 2, 1975, pp. 52-57.)

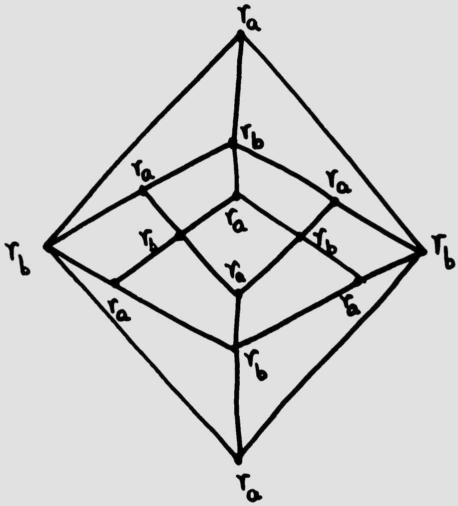
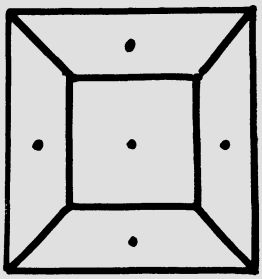
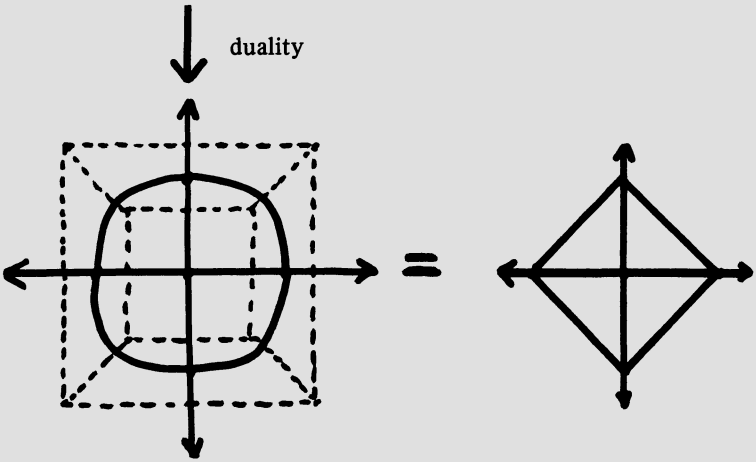


FIGURE 8-5 Schlegel diagram of a rhombohedral dodecahedron $r_a=3$, $r_b=4$, $n=4$



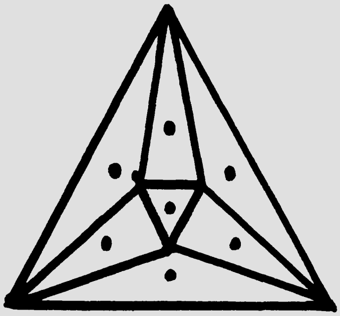
Schlegel Diagram of the cube, with five vertices of its dual shown. The sixth vertex is not visible here.



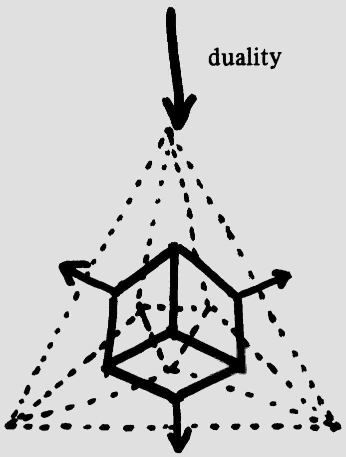
Dual of the cube: the arrows indicate convergence of edges to the invisible sixth vertex.

Dual Schlegel Diagram of the octahedron

FIGURE 8-6



Schlegel Diagram of the octahedron, showing seven of the eight vertices of its dual



Dual Schlegel Diagram of the cube

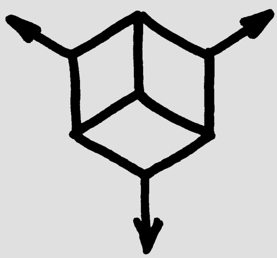
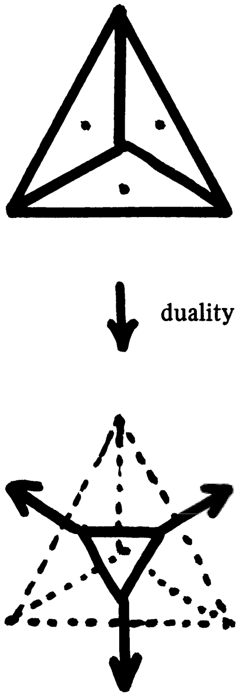


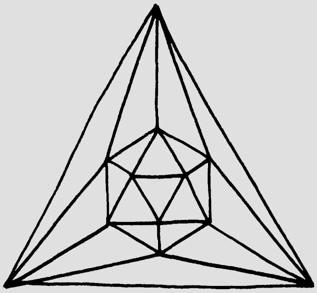
FIGURE 8-7



Schlegel Diagram of the tetrahedron showing three of the four vertices of its dual

Dual Schlegel Diagram of the tetrahedron

FIGURE 8-8



9

Regular Structures

In Chapter 5 we discussed *statistical symmetry,* the fact that the *average* values of the valencies of structures obey some very rigid restrictions. The present chapter is the very antithesis of Chapter 5, for here we examine structures in which all elements of any one dimensionality have identical sets of valencies. Here the average values become *the* values of the valencies. The basic equations (4-11) through (4-14), and (3-4) and (3-5) are then

$$\frac{1}{p} - \frac{1}{s} + \frac{1}{n} - \frac{1}{l} = 0 \qquad (9\text{-}1)$$

$$\frac{1}{2}r = p/s \qquad (9\text{-}2)$$

$$\frac{1}{2}k = l/n \qquad (9\text{-}3)$$

$$p/q = l/m \qquad (9\text{-}4)$$

$$m - l + k = 2 \qquad (3\text{-}4)$$

$$r - p + q = 2 \qquad (3\text{-}5)$$

First we consider two-dimensional structures: $s = 2, p = r, q = 2$.

$$\therefore \frac{1}{r} + \frac{1}{n} = \frac{1}{2} + \frac{1}{l} \qquad (9\text{-}5)$$

For $r = 2$, $n = l$; hence from equation (9-3), $k = 2$; from equation (3-4), $m = l$; and from equation (9-4), $p = q = 2$. The corresponding structure is a polygon having l edges and l vertices, and two faces (top and bottom), as shown in Fig. 9-1. Each vertex has a valency 2 toward edges, faces, and cells, and the cell is lens-like, having a top and bottom face, l edges, and l vertices. Having *two* faces, it is called a *dihedron.*

For $r = 3$,

$$n = \frac{6l}{l+6}, \qquad l = \frac{6n}{6-n}$$

Since n must be an integer, the minimal value of l is 3.

For $r = 3$, $l = 3$, we find from the above-listed equations: $n = 2$, $k = l = 3$, $m = 2$. The corresponding structure (Fig. 9-2) has two vertices that are joined to each other by three edges and three faces. Such a structure has physical reality only because we admit curved edges and faces; since it follows organically from our basic equations, it is important and should not be overlooked. The faces are digons ($n = 2$). The structure of Fig. 9-2 is called a *digonal trihedron.* We shall see when we discuss truncations and stellations that such structures play a fundamental role.

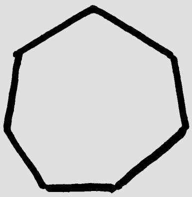


FIGURE 9-1 Structure having r=2, s=2: b=gonal dihedron

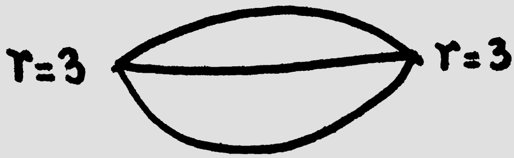


FIGURE 9-2 Structure having r=3, s=2: digonal trihedron

The next integral value of n occurs when $l = 6$: $l = 6$, $n = 3$, $k = 4$, $m = 4$. The corresponding structure is the (triangular) tetrahedron (Fig. 9-3).

Next for $r = 3$, we have $l = 12$, $n = 4$, $k = 6$, $m = 8$. The corresponding structure is the hexahedron, of which the cube is a particular example (Fig. 9-4). The next integral value of n occurs when $l = 30$, $n = 5$, $k = 12$, $m = 20$. The corresponding structure is the pentagonal dodecahedron (Fig. 9-5). Finally, the value n - 6 is not

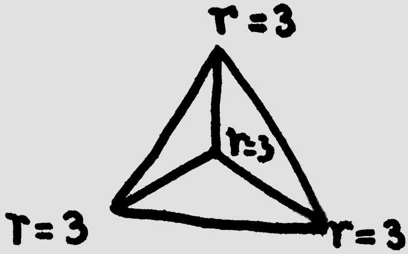


FIGURE 9-3 Tetrahedron (trigonal)

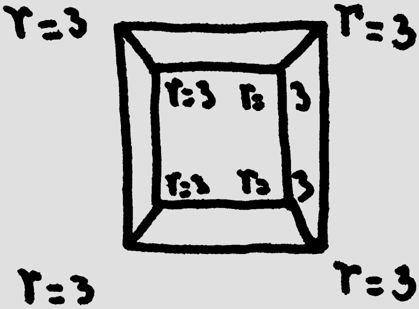


FIGURE 9-4 Cube

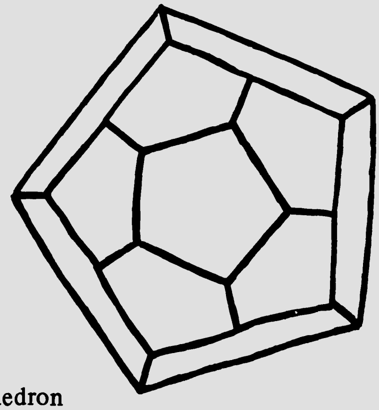


FIGURE 9-5
Pentagonal dodecahedron

reached until $l = \infty$. The resulting structure extends indefinitely and is illustrated by an infinitely extended tessellation of a plane (Fig. 9-6). This exhausts all regular structures having $s = 2$, $r = 3$.

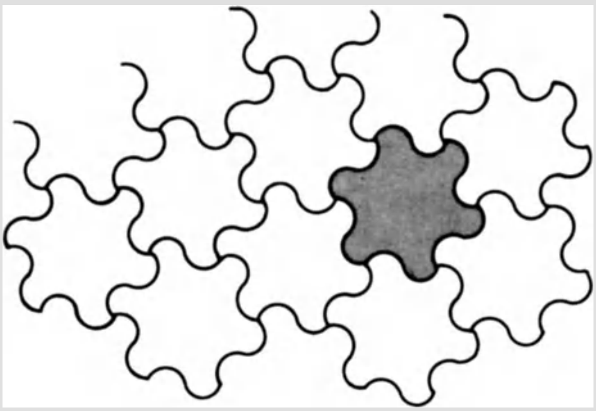


FIGURE 9-6 Hexagonal tessellation

Next we turn to $s = 2, r = 4$. From equation (9-5) we have

$$n = \frac{4l}{l + 4}; \quad l = \frac{4n}{4 - n}$$

The only possible values of n are $n = 2$, 3, or 4. The digonal structure, $n = 2$, has $l = 4$, $k = 4$, $m = 2$: it is a *digonal* tetrahedron (Fig. 9-7). The value $n = 3$ gives $l = 12$, $k = 8$, $m = 6$: the octahedron (Fig. 9-8). Finally, $n = 4$ occurs when $l = \infty$: this structure is a square or quadrilateral tessellation (Fig. 9-9).

FIGURE 9-7
Digonal tetrahedron

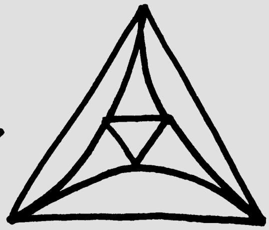


FIGURE 9-8
Octahedron

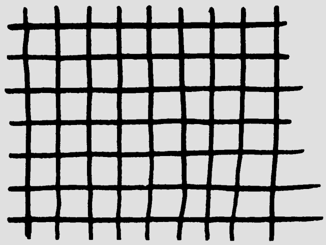


FIGURE 9-9
Quadrilateral tessellation

For $r = 5$,

$$n = \frac{10l}{3l + 10}, \quad l = \frac{10n}{10 - 3n}$$

Again there is a digonal structure: $n = 2$, $l = 5$, $k = 5$, $m = 2$, which does not need explicit illustration: it must be clear by now that (cf. equation 9-5) when $n = 2$ there exists a whole *family* of digonal polyhedra having $m = 2$, $l = k$. However, there is *no* possibility of an infinite tessellation, for $l = \infty$ does not yield an integral value of n. Thus *no* infinite tessellation with equivalent 5-valent vertices and mutually equivalent faces is possible, *regardless* of the exact shape of the face. This observation is very important, for it has been shown[1] that fivefold *symmetry* cannot exist in a periodically repeating *planar* pattern. However, it is apparent that, for instance in much Islamic ornament, there have been attempts to distort pentagonal faces to conform to the symmetry constraints. We have shown here that

[1] Cf., for instance, Arthur L. Loeb: *Color and Symmetry* (Wiley, New York, 1971), p. 33, theorem 10.

there is a *connectivity* restriction on the number 5 in an infinitely extended tessellation as well as the *symmetry* restriction.

Besides the digonal pentahedron there is only a single other regular two-dimensional structure having $r = 5$—namely, the one having $n = 3, l = 30, k = 20, m = 12$: the icosahedron (Fig. 9-10).

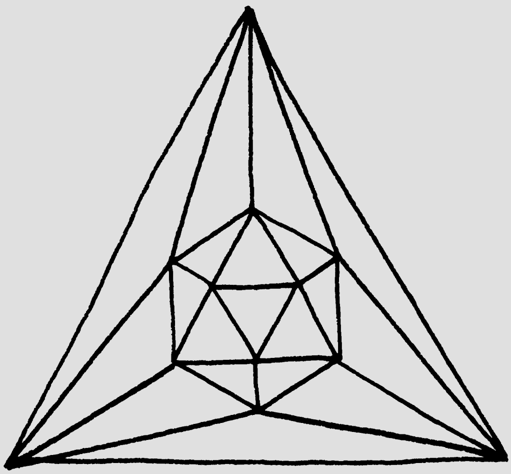


FIGURE 9-10 The Icosahedron

Finally, there is a single structure having $r = 6$ besides the ubiquitous digonal one. When $r = 6$ in equation (9-5),

$$n = \frac{3l}{l + 3}, \qquad l = \frac{3n}{3 - n}$$

The value $n = 2$ yields the digonal hexahedron, and $n = 3$ yields a tessellation by triangles (Fig. 9-11). Thus we exhaust the enumeration of regular two-dimensional structures, which are listed in Table 9-1. In this table there are three entries having $r = n$: the digonal dihedron ($n = r = 2$), the (trigonal) tetrahedron ($n = r = 3$), and the square tessellation ($n = r = 4$). These entries represent the three self-duals among regular two-dimensional structures. All other structures represented in Table 9-1 are paired duals: interchanging the values of r and n for any entry locates its dual in the table. The digonal polyhedra are the duals of the polygonal dihedra (two-sided polygons), and only these allow *any* integral value of r, respectively n.

The rows $n = 3$, 4, and 6 and the columns $r = 3$, 4, and 6 in Table 9-1 terminate with a tessellation of infinite extent. The row $n = 5$

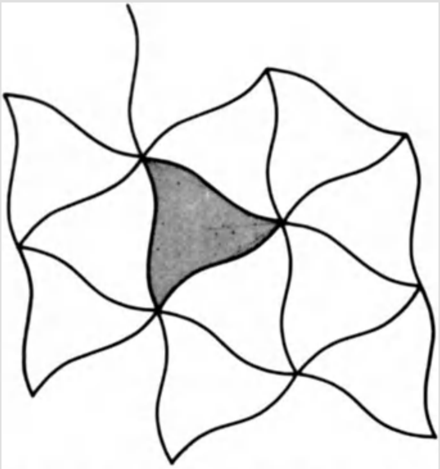


FIGURE 9-11 Triangular tessellation

TABLE 9-1
Regular Two-Dimensional Structures

			r →			
	2	3	4	5	6	r ...
2	Digonal dihedron	Digonal trihedron	Digonal tetra-hedron	Digonal penta-hedron	Digonal hexahedron	Digonal r-hedron
3	Triangular dihedron	(Triangular) tetrahedron	Octa-hedron	Icosa-hedron	Triangular tessellation	
n ↓ 4	Quadri-lateral dihedron	Hexahe-dron	Square tessellation			
5	Pentagonal dihedron	Pentagonal dodecahe-dron				
6 . . .	Hexagonal dihedron	Hexagonal tessellation				
n	n-gonal dihedron					

and the column $r = 5$ end, respectively, in the dodecahedron and icosahedron, as no infinite tessellations are possible for these values.

Next we consider regular structures having $s = 3$. These are *three-dimensional*, and, since the cells must themselves be regular, we are limited to those cells just found to constitute the regular two-dimensional structures (cf. Table 9-1).

From equation (9-2), making $s = 3$: $r = (^2/_3)p$. Hence from equation (3-5),

$$q = 2 + \frac{1}{3}p \tag{9-6}$$

Consider first the digonal polyhedra. These all have $n = 2$, hence (equation 9-3) $k = l$, and (equation 3-4) $m = 2$. Therefore (equation 9-4), $p = \frac{1}{2}lq$, and hence (equation 9-6)

$$q = \frac{12}{6 - l}, \quad p = \frac{6l}{6 - l}, \quad r = \frac{4l}{6 - l}$$

These expressions limit l to values less than or equal to 6. When $l = 2$: $p = 3$, $q = 3$, $r = 2$. These values are exemplified by a sphere having vertices at north and south poles, which is bisected by a *plane* containing two meridians (the two edges). There are *three* cells: the two hemispheres, and the "outside world." Three faces meet at either edge: the two hemispherical surfaces and the partition.

For $l = 3$ (s still equals 3, of course), $q = 4$, $p = 6$, $r = 4$. This structure is exemplified by a sphere or pod (Fig. 9-12) subdivided

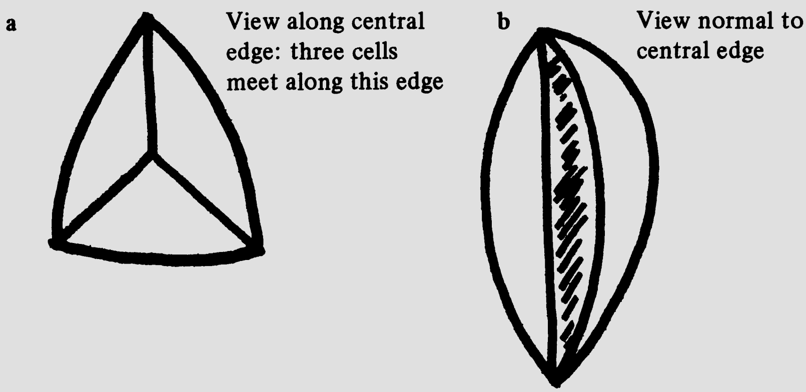


FIGURE 9-12 Structure having s=3, r=4, l=3, m=2, k=3, n=2

into three equivalent cells by three meridians and a north–south axis. At each pole four edges meet ($r = 4$); at each edge three cells meet: for the meridians these are the "outside world" and two internal cells, and for the north–south axis there are three internal cells ($s = 3$). At each pole also, *four* cells meet ($q = 4$), of which one is the "outside world," and six faces meet (p - 6), of which three are portions of the spherical surface, and three are internal partitions. Figure 9-12*a* could be considered here to be a cross section of such a structure perpendicular to the north-south axis, where the *edges* of Fig. 9-12a are now the *traces* of the faces on the cross section.

For $l = 4, s = 3$: $q = 6, p = 12, r = 8$. This structure has again two poles, and internal partitions whose cross sections are illustrated by Fig. 9-4. If we recall that Figs. 9-3 and 9-4 are actually Schlegel diagrams of a tetrahedron and a cube, respectively, then we may look at the present structures as Schlegel diagrams in *three* dimensions of four-dimensional regular structures. Just as Fig. 9-4 represents a cube in perspective distortion, so the cells of the present structures, which in *three* dimensions could not be made geometrically congruent, can be considered perspective distortions of what are in *four* dimensions mutually congruent cells.

For $l = 5, q = 12, p = 30, r = 20$ Again this is a "pod"-like structure, with internal faces whose cross section looks like Fig. 9-5, a sort of four-dimensional hyperpentagonal dodecahedron, a three-dimensional structure in a four-dimensional space, analogous to the two-dimensional polyhedron in three-dimensional space.

Finally, for $l = 6, q = \infty, p = \infty, r = \infty$. The structure still has a north and a south pole which are the vertices for every cell, but from these vertices infinitely many edges, faces, and cells, of generally *finite* size each, emanate, yielding in cross section the infinite hexagonal tessellation of Fig. 9-6. This exhausts the regular structures having $s = 3$ and digonal polyhedra as cells.

Consulting Table 9-1, we consider next the regular three-dimensional structures having as cells the polygonal dihedra in the column $r = 2$. These cells are characterized by the parameters $n = l = m$, and $k = 2$. Therefore (since $s = 3$ still), from equation (9-4), $p = q$; and from equation (9-2), $r = 2p/3$. Substitution in equation (3-5) yields $p = 3, q = 3, r = 2$. These parameters correspond to an l-gon having front and back faces ("lens-like") "floating" on an infinitely extended third face, or to a lens-like l-gon having a front, *middle,* and back face. Whenever $n = l$, equation (9-1) demands $p = s$: for any value of s there will be this type of internal face joining the external polygon. These cases are fairly trivial; we need not further consider the polygonal dihedral cells for additional values of s.

We are then left with a finite set of regular cells. For the (triangular) tetrahedron, $k = 4$, $l = 6$, $m = 4$, $n = 3$. Therefore, from equation (9-4), $p = 3/2\ q$; from equation (9-2), $r = q$; and from equation (3-5), $p = 2q - 2$. Therefore $q = r = 4$, $p = 6$. The corresponding structure is a tetrahedron subdivided into four cells, each a tetrahedron, which share a vertex at the center of the original tetrahedron (Fig. 9-13). For the cube, $k = 6$, $l = 12$, $m = 8$. Hence

$$p = \frac{3}{2}q; \quad r = \frac{2}{3}p = q; \quad \therefore q = 4, p = 6, r = 4$$

These vertex parameters correspond to the structure illustrated in Fig. 9-14, a Schlegel diagram in three dimensions of a hypercube.

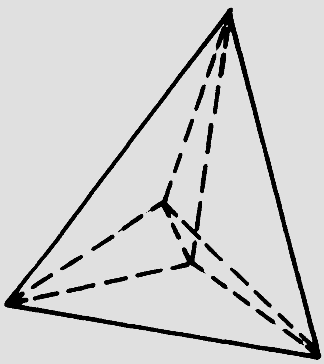


FIGURE 9-13 Hypertetrahedron

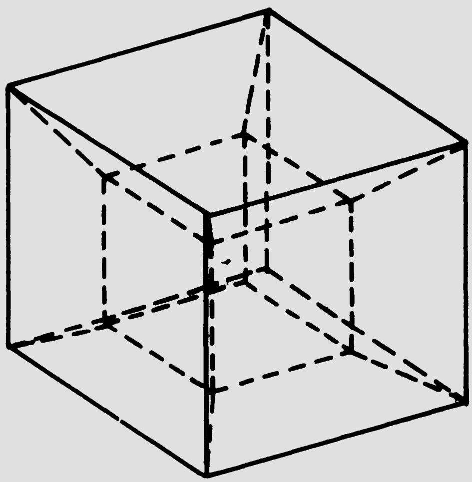


FIGURE 9-14 Hypercube

It should be noted in Figs. 9-13 and 9-14 that the outside surface of the structure is equivalent to each of the cells: in Fig. 9-13 the outside surface is a tetrahedron, in Fig. 9-14 a cube. Although this observation is remarkable, it should not be surprising, for we have always considered the outside world as a cell equivalent to the internal cells, so that *all* faces, "internal" as well as "external," should be *interfaces* between cells. Therefore the so-called outside world should have valencies identical to those of the so-called internal cells.

When a wire frame constituting the edges of one of our regular polyhedra is dipped in a soap solution, one of the configurations discussed here is apt to develop, with internal cells reflecting the outside shape. Soap films invariably occur in configurations having $s = 3$, l $q = 4$, $r = 4$, $p = 6$, these being the simplest (smallest) three-dimensional configurations.

For the octahedron, $k = 8$, $l = 12$, $m = 6$. Hence $q = 6$, $r = 8$, $p = 12$. This structure corresponds to a hyperoctahedron; this and similar regular structures in four-dimensional space are discussed by Coxeter.[2] A model of a hyperoctahedron having 24 vertices, 96 edges, 96 faces, and 24 cells, constructed by Mabel Liang and Tad Paul, is shown in Fig. 9-15. Since the number of vertices equals the number of cells, and the number of edges equals the number of faces, this structure is self-dual.

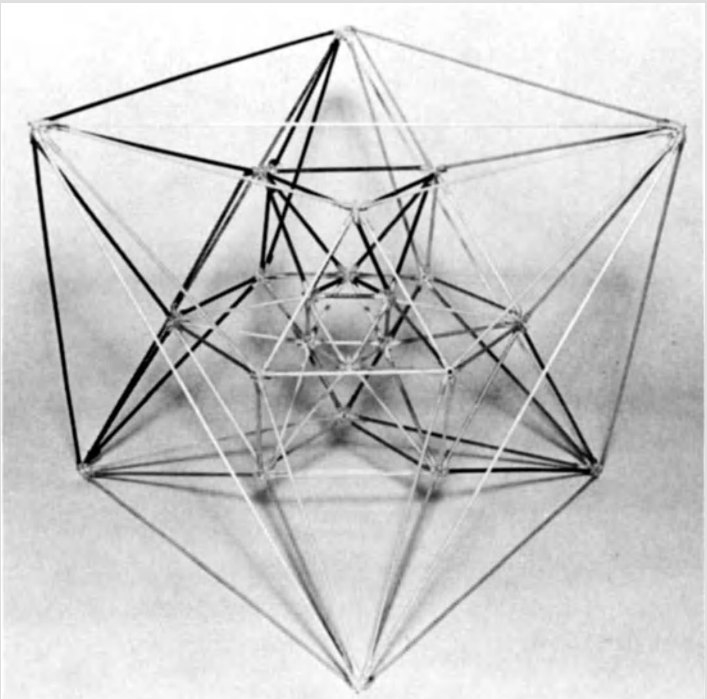


FIGURE 9-15 Hyperoctahedron (Model by Mabel Liang and Tad Paul; photograph by C. Todd Stuart)

[2] H. S. M. Coxeter: *Introduction to Geometry* (Wiley, New York, 1961), Section 22.3.

It is obvious that, given a set of *cell* parameters, k, l, m, and n, one can use equations (3-5), (9-4), and (9-2) to find the *vertex*, or *connection* parameters p, q, r, and s. [Remember that equation (9-1) is not independent of the others.] There is thus one *free* parameter, for instance s. For *every* value of s and for every type of cell (i.e., set of values k, l, m, and n) one may go through a process analogous to that which we traversed for $s = 3$. For instance, $s = 4$ yields an interesting and significant structure for the cubic cell: $k = 6$, $l = 12$, $m = 8$, $n = 4$; hence $p = 12$, $q = 8$, $r = 6$. These values corresponds to an infinite array of stacked cubes, superficially analogous to a tessellation. There is, however, a fundamental distinction between two- and three-dimensional systems. The solutions of equations (9-5) ff. are all enumerated in Table 9-1, a finite set of two-dimensional regular structures, terminating in the tessellations when $l = \infty$. Equation (9-1), on the other hand, does not contain any parameter explicitly expressing the *number* of vertices, edges, faces, or cells: it is homogeneous in the valencies. In three dimensions the number of valencies exceeds the number of constraining equations by *three.* Thus the three-dimensional structures differ fundamentally from the two-dimensional ones: the former have much greater freedom of choice, and are not exhaustively enumerable in a manner analogous to that used for two-dimensional structures.

Accordingly, we shall exhaustively enumerate all two-dimensional *semi*regular structures, which may serve as cells in three-dimensional structures. Once these cells are found, the continuity equations (4-12), (4-13), and (4-14), together with equations (3-4) and (3-5), will then give all possible connection valencies p, q, r, and s for any given set of all valencies k, l, m, and n. By the duality argument (Chapter 7) one may analogously start with a given set of connection valencies and find the corresponding constraints on cells that may be so connected.

To this purpose equation (4-11) may be put in parametric form:

$$\frac{1}{\bar{s}} - \frac{1}{\bar{p}} = \frac{1}{f} \tag{4-11a}$$

$$\frac{1}{\bar{n}} - \frac{1}{\bar{l}} = \frac{1}{f} \tag{4-11b}$$

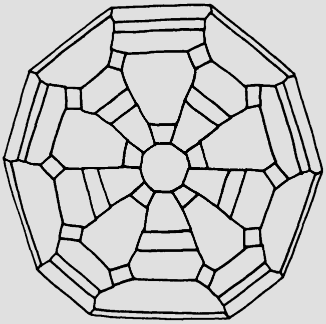
The former equation deals *only* with connection valencies and f, the latter with cell parameters and f. When equations (4-11) and (4-13) are substituted:

$$\bar{p} = \frac{1}{2} f (\bar{r} - 2) \tag{9-6}$$

$$\bar{l} = \frac{1}{2} f (\bar{k} - 2) \tag{9-7}$$

Here the valencies of *any* cell permit us to find f from equation (9-7), and, using the resulting value of f, to relate p to r by equation (9-6). Conversely, one may find f for a given connection from equation (9-6), and subsequently test any cell for such connectivity with the aid of equation (9-7).

The problem of finding (topologically) regular three-dimensional structures should not be confused with the related one of finding mutually congruent cells that together fill all of space (Chapter 17). These space-filling cells need not be regular polyhedra; neither are their vertices mutually equivalent. The cube only appears to be capable of functioning as a cell for a *space filling* as well as *regular* structure.



10

Truncation and Stellation

In Chapter 7 we discussed duality relations, in which elements of different dimensionalities are interchanged, subject to the constraints of the Euler-Schlaefli relation. In particular, we discussed in Chapter 7 the interchange of vertices and faces.

In the present chapter we replace *edges* by *faces* ("edge truncation") and by *vertices* ("edge stellation"). The former transformation is illustrated in Fig. 10-1: it is observed that, as a result of edge truncation, there are new faces not only in the place of old *edges* but also in the place of the old *vertices.* This edge truncation process is a very general one: special cases to be considered are those for which the new edges marked *a, b,* and *c* in Fig. 10-1 vanish in various combinations. In particular, let us consider a *regular* structure, in which *every* edge is truncated equivalently. If the new edges labeled *c* vanish, then the original *r*-valent vertices are replaced by *r*-valent faces, while the original *n*-valent faces remain *n*-valent. The resulting structures are discussed in detail in Chapter 12, an exhaustive enumeration of all semiregular structures. If, in addition to the *c* edges, the ones marked *a* vanish as well, then the original *n*-valent faces turn into *n*-valent vertices at which the new edges marked *b* converge, each *b* edge replacing an original edge. The result is the *dual* of the original structure: duality is thus seen to follow as a special case of edge truncation. (The structures resulting when *c*

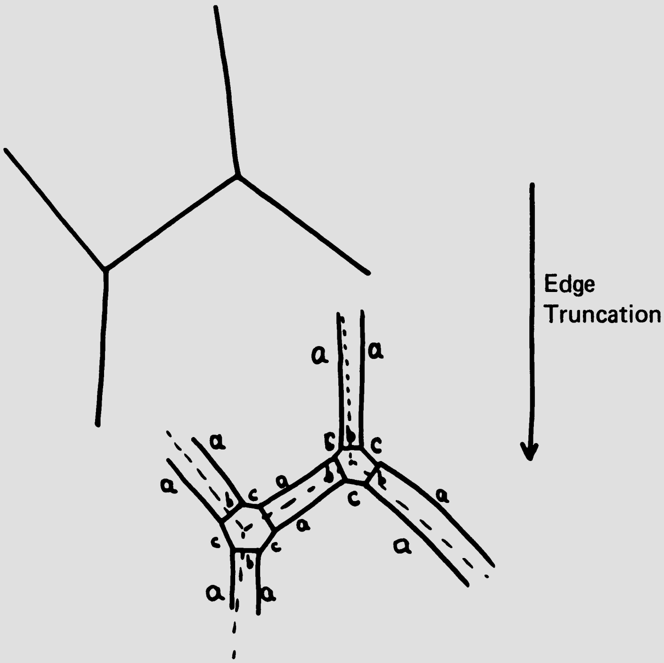


FIGURE 10-1 Edge truncation

edges but *not a* edges vanish will be shown to be combinations of superimposed dual pairs.)

Figure 10-2 shows *edge stellation.* Here a vertex is introduced on an edge, from which new edges emerge, converging toward new vertices. Since the *directions* of these new edges may be chosen at will, it is possible, as a special case, to choose them such that the faces adjoining an original vertex coalesce into a single face, so that again the dual is produced as a special case.

For both edge truncation and edge stellation we observe that *each* original edge gives rise, in the most general case, to *four* new edges, which degenerate into a single new one in the special case of forming a dual. An r-valent vertex in the original structure produces in general $2r$ edges: in the case of *truncation* it produces a $2r$-valent face, whereas in the case of *stellation* it produces a $2r$-valent vertex. Conversely, an n-valent face of the original structure is turned by *edge truncation* into a $2n$-valent face, whereas *edge stellation* turns it into a $2n$-valent vertex. Note, furthermore, that *edge truncation* generally produces *trivalent vertices,* while *edge stellation* produces *triangular faces.*

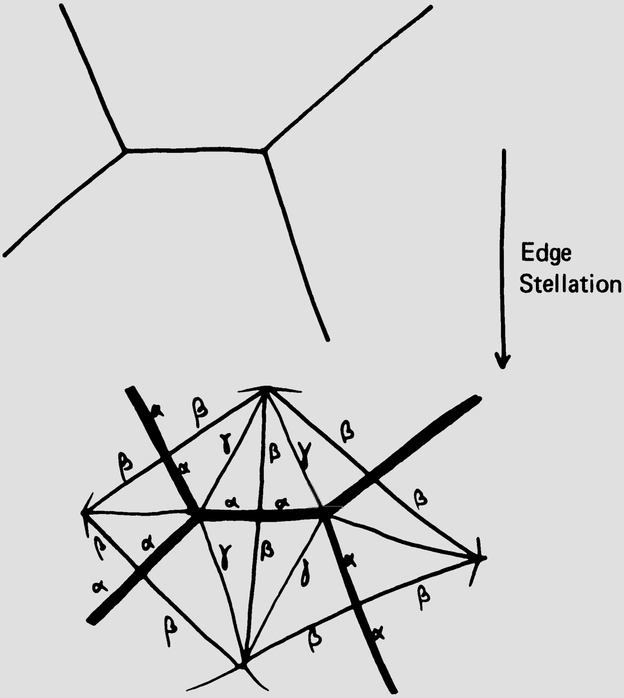


FIGURE 10-2 Edge stellation

If we truncate the edges of a regular structure having V vertices, E edges, and F faces, we can find the parameters of the resulting structures as follows. Figure 10-1 illustrates that each original edge yields *four* new vertices, also *two* edges of the type labeled a and $2r$ edges of the types labeled b and c. Furthermore, *each* vertex, edge, and face of the original structure yields a *face* in the truncated structure. Therefore, if V_{Et}, E_{Et}, and F_{Et} are the numbers of vertices, edges, and faces, respectively, of the truncated structure:

$$V_{Et} = 4E; \quad E_{Et} = 2E + 2rV; \quad F_{Et} = F + E + V$$

From equation (4-4): $2rV = 4E$. From equation (4-6) and the fact that for two-dimensional structures (tessellations or polyhedra) $s = 2$: $F = \frac{2}{n}E$. Hence

$$E_{Et} = 6E \quad \text{and} \quad F_{Et} = \left(\frac{2}{n} + 1 + \frac{2}{r}\right)E$$

From equation (9-5): $\frac{2}{n} + \frac{2}{r} = 1 + \frac{2}{E}$.

$$\therefore F_{Et} = 2(E + 1)$$

Summarizing, we find the following relations between the number of elements of the general edge-truncated structures and the number of edges of the original regular structure:

$$V_{Et} = 4E \tag{10-1}$$

$$E_{Et} = 6E \tag{10-2}$$

$$F_{Et} = 2(E + 1) \tag{10-3}$$

Table 10-1 lists these parameters for the generally truncated forms derived from regular polyhedra.(Figs. 10-3, 10-4, and 10-5). It is

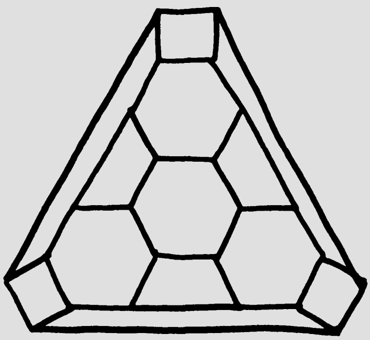


FIGURE 10-3 Schlegel diagram of the edge-truncated tetrahedron (equivalent to truncated octahedron)

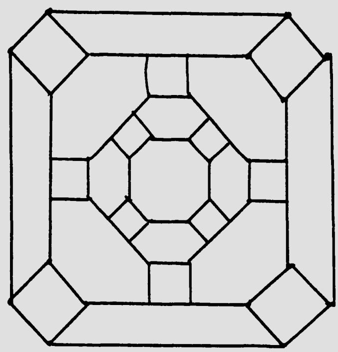


FIGURE 10-4 Schlegel diagram of the edge-truncated octahedron or cube (great rhombicuboctahedron)

observed that the five regular polyhedra yield *three* truncated forms, because dual regular pairs yield the same generally truncated form: the octahedron/cube pair yields a solid having eight hexagonal, twelve quadrilateral, and six octagonal faces (Fig. 10-4, the great rhombicuboctahedron), and the icosahedron/dodecahedron pair yields a solid having twenty hexagonal, thirty quadrilateral, and twelve decagonal faces (Fig. 10-5, the great rhombicosidodecahedron).

TABLE 10-1
Parameters of Edge-Truncated Regular Solids

Regular solid	*E*	V_{Et}	E_{Et}	F_{Et}	*Specification of Faces*						*Name of truncated solid*
					Corresponding to original faces		*Corresponding to original edges*		*Corresponding to original vertices*		
					Number	*Valency* ($2n$)	*Number*	*Valency*	*Number*	*Valency* ($2r$)	
Tetrahedron	6	24	36	14	4	6	6	4	4	6	Kelvin solid (vertex-truncated octahedron)
Octahedron	12	48	72	26	8	6	12	4	6	8	Great rhombicuboctahedron
Cube					6	8	12	4	8	6	
Icosahedron	30	120	180	62	20	6	30	4	12	10	Great rhombicosidodecahedron
Pentagonal dodecahedron					12	10	30	4	20	6	

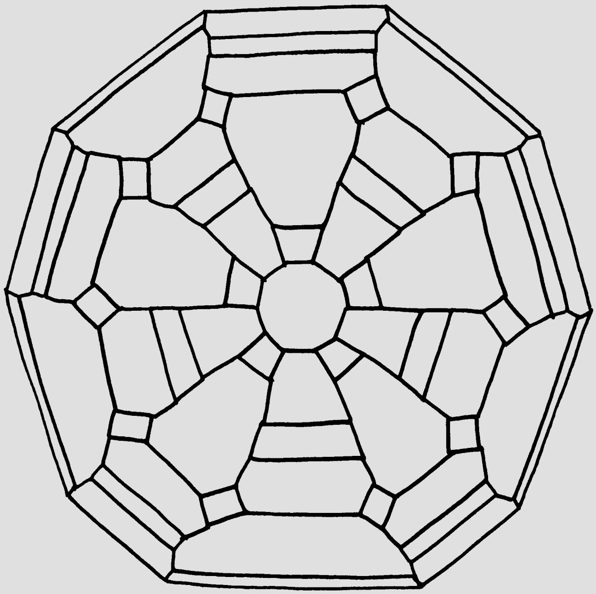


FIGURE 10-5 Schlegel diagram of the edge-truncated icosahedron or dodecahedron (great rhombicosidodecahedron)

These truncated structures all have mutually equivalent trivalent vertices, but three different types of faces. They are called *semiregular* solids. Such solids have mutually equivalent vertices but not faces, *or* mutually equivalent faces but not vertices. Some consider only those having equivalent vertices *or* those having equivalent faces semiregular. However, the symmetry of the *structure of structures* makes such a bias irrational; either will here be called semiregular.

Besides the so-called Platonic solids (tetrahedron, octahedron, cube, icosahedron, and pentagonal dodecahedron), we found (cf. Table 9-1) the following regular two-dimensional structures: the digonal polyhedra, having $n = 2$; the polygonal dihedra, having $r = 2$; and the tessellations, having $l = \infty$. The digonal polyhedra have two r-fold vertices, and $E = r$. When these are truncated, prisms are formed, whose structural parameters are (cf. equations 10-1, 10-2, 10-3)

$$V_{Et} = 4r, \qquad E_{Et} = 6r, \qquad F_{Et} = 2(r + 1)$$

Of the faces, two are r-gons, and the remaining $2r$ faces are quadrilateral. Edge truncation of the polygonal dihedra similarly produces prisms, as was to be expected, for we have seen above that dual regular structures produce the same semiregular structure by edge truncation (Fig. 10-6). In particular, the digonal dihedron yields a hexahedron (cube): when $r = 2$, $V_{Et} = 8$, $E_{Et} = 12$, $F_{Et} = 6$.

Finally, edge truncations of the three regular tessellations yield the two semiregular tessellations shown in Figs. 10-7 and 10-8. The triangular and hexagonal tessellations, being each other's duals, yield the same truncated structure, and the square tessellation, being self-dual, yields a single structure by itself.

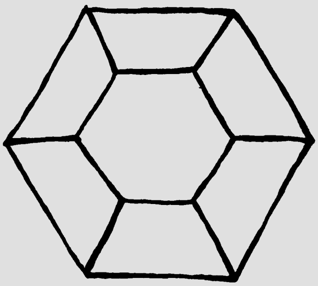


FIGURE 10-6 Prism resulting from the edge truncation of a diagonal trihedron and of a trigonal dihedron

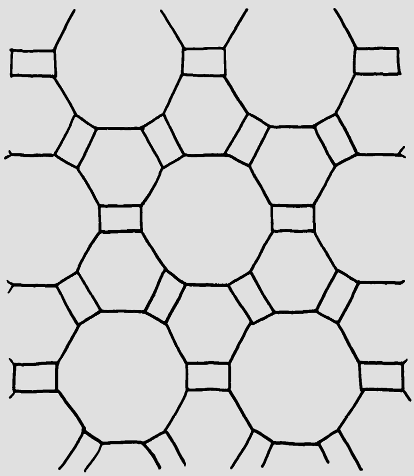


FIGURE 10-7 Edge-truncated triangular or hexagonal tessellation

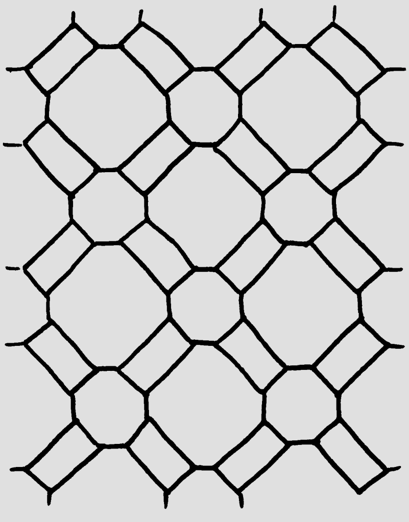


FIGURE 10-8 Edge-truncated square tessellation

(These edge-truncated structures have a special significance for the crystallographer: their vertices represent "general positions." The number 48, equaling the number of vertices of an edge-truncated cube or octahedron, also equals the multiplicity of the most general *point* or *lattice complex* in the primitive cubic system, illustrated in Fig. 10-9. It is significant, however, that the crystallographic point complexes derive from symmetry considerations, whereas our results derive ultimately from Euler's topological law, and therefore are not limited to cubic symmetry. The structures about to be discussed are analogously related to more special point complexes.)

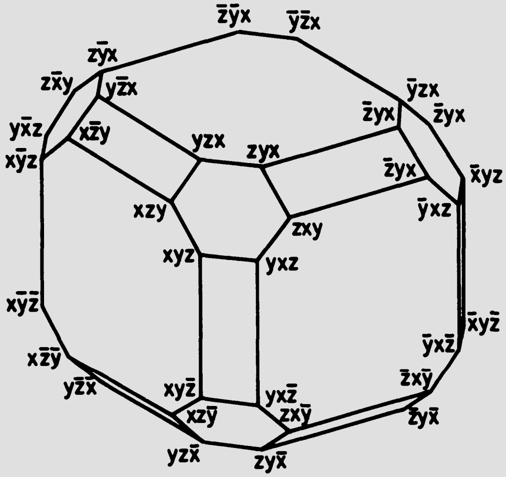


FIGURE 10-9 Edge-truncated cube or octahedron, showing the coordinates of a general point complex in the crystallographic cubic lattice

In Fig. 10-1 we labeled the newly created edges resulting from general edge truncation *a, b,* and *c.* We noted that when the edges labeled *a* and *c* all vanish, so that only the *b* edges remain, the dual of the original structure is formed.

We shall now investigate other special combinations of *a, b,* and *c.* When the *c* edges vanish, we find a group of structures which will be discussed in Chapter 12. When the *b* edges vanish, the edges of the the original structure are partially intact, and the vertices of the original structure are replaced by faces of the same edge valency. On the other hand, when the *a* edges vanish, the edges of the *dual* of the original structure are preserved, and instead of the vertices of the dual structure there are faces of equal edge valencies. Accordingly, we call these special cases of the general edge truncation *vertex trun-*

cations: when $b = 0$, the original structure is vertex-truncated, whereas when $a = 0$, it is the dual of the original structure that is vertex-truncated.

The concept of vertex truncation is a relatively familiar one; that of the general edge truncation is not. It is interesting, however, that the formation of the dual as well as the vertex truncations of both dual structures are special cases of edge truncation. These relationships are particularly important in the field of computer graphics, where whole families of regular and semiregular structures can be generated by means of the relations derived here.

When both b and c vanish, the original structure remains unaltered. When both a and b vanish, every edge of the original structure has vanished: both the original structure and its dual have had their vertices truncated, with the result that every original edge is replaced by a quadrivalent vertex! We call this type of truncation *degenerate* truncation, because the two trivalent vertices which appear on each edge when vertices are truncated here degenerate into a single quadrivalent vertex. The special truncations are summarized in Fig. 10-10.

Vertex truncation of a regular structure yields the following parameters, which are denoted V_{Vt}, E_{Vt}, and F_{Vt}:

$$V_{Vt} = 2E; \quad E_{Vt} = E + rV; \quad F_{Vt} = F + V$$

From equation (4-4) again: $rV = 2E$, hence $E_V = 3E$. Furthermore

$$F_{Vt} = F + \frac{2}{r}E = \left(\frac{2}{n} + \frac{2}{r}\right)E = E + 2$$

Accordingly:

$$V_{Vt} = 2E \tag{10-4}$$

$$E_{Vt} = 3E \tag{10-5}$$

$$F_{Vt} = E + 2 \tag{10-6}$$

Since dual structures have equal numbers of edges, these expressions hold for any regular structure as well as for its dual. However, the two structures resulting from vertex truncations of duals, unlike the results of edge truncations, are not identical. If a regular structure having r-valent vertices and n-valent faces is vertex-truncated, the

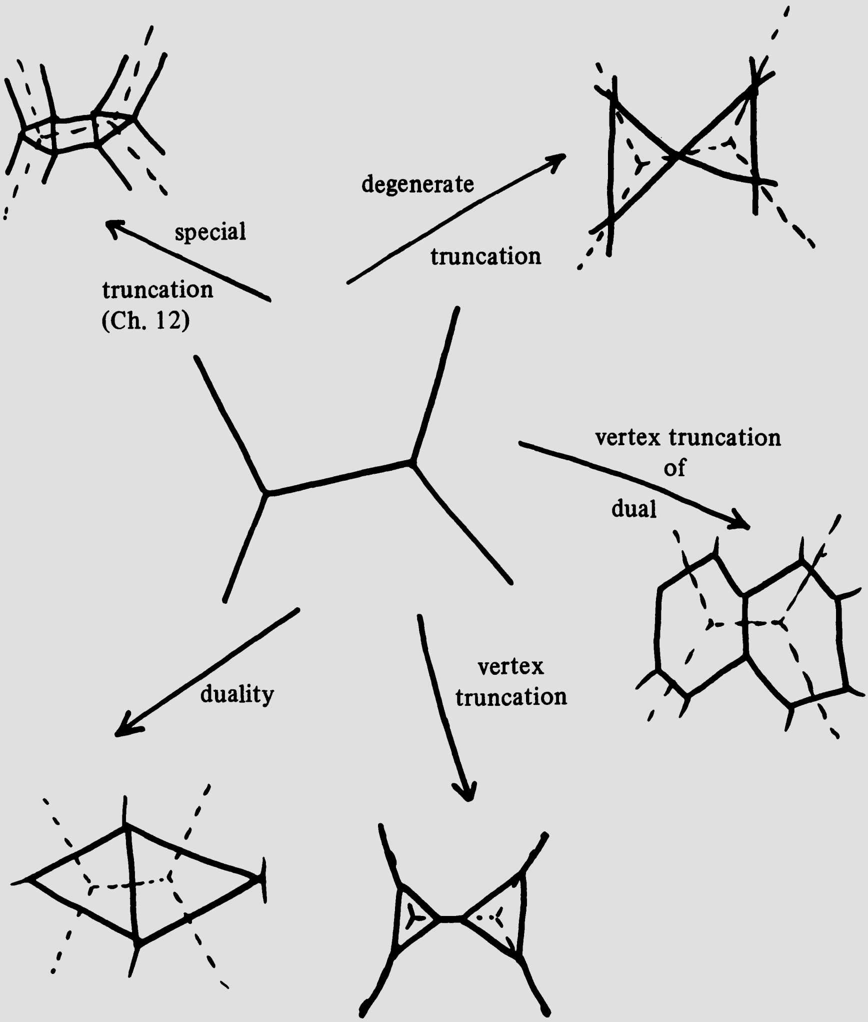


FIGURE 10-10 Summary of special truncations

result is a structure having r-valent as well as $2n$-valent faces, whereas its dual would give $2r$-valent as well as n-valent faces. For instance, the *truncated cube* has *eight triangular* as well as *six octagonal* faces, whereas the *truncated octahedron* has *eight hexagonal* as well as *six square* faces. Table 10-2 lists the results of vertex truncation of regular polyhedra. (Note again that truncation of the icosahedron yields exactly the twelve permitted pentagons combined with hexagons; cf. Chapter 6.)

TABLE 10-2
Parameters of Vertex-Truncated Regular Solids

Regular solid	*E*	V_{Vt}	E_{Vt}	F_{Vt}	*Specification of Faces* — *Corresponding to original faces*		*Corresponding to original vertices*		*Name of truncated solid*
					Number	*Valency* ($2n$)	*Number*	*Valency* (r)	
Tetrahedron	6	12	18	8	4	6	4	3	Truncated tetrahedron ("Laves polyhedron")
Octahedron	12	24	36	14	8	6	6	4	Truncated octahedron ("Kelvin solid")
Cube	12	24	36	14	6	8	8	3	Truncated cube
Icosahedron	30	60	90	32	20	6	12	5	Truncated icosahedron
Pentagonal dodecahedron	30	60	90	32	12	10	20	3	Truncated dodecahedron

Vertex truncation of the digonal r-hedron yields a prism having two r-gonal and r quadrilateral faces, that of the n-gonal dihedron a structure illustrated in Fig. 10-11, resembling a lens having facets cut along its circumference. Figures 10-12, 10-13, and 10-14 represent the vertex-truncated tessellations, Figures 10-15 through 10-19 the vertex-truncated Platonic solids. (Note that the truncation of the triangular tessellation results in a hexagonal tessellation, which is topologically regular.)

Degenerate truncation yields a family of structures whose quadrivalent *vertices* correspond to the *edges* of the parent structures: their parameters are indicated by V_{dt}, E_{dt}, and F_{dt}:

$$V_{dt} = E; \quad E_{dt} = rV; \quad F_{dt} = F + V$$

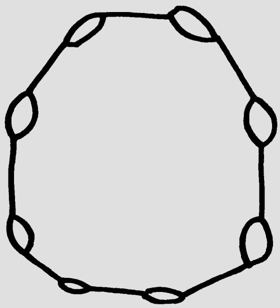


FIGURE 10-11 Vertex-truncated r-gonal dihedron

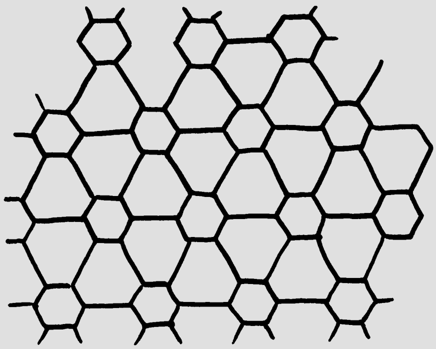


FIGURE 10-12 Vertex-truncated triangular tessellation (hexagonal tessellation)

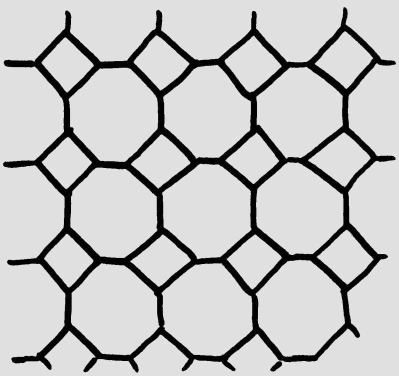


FIGURE 10-13 Vertex-truncated square tessellation (octagon-square tessellation)

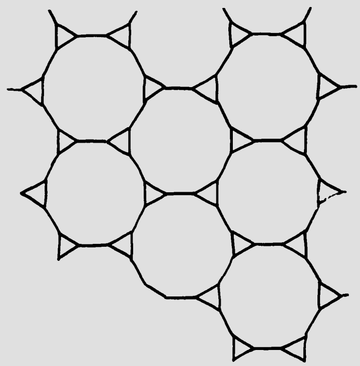


FIGURE 10-14 Vertex-truncated hexagonal tessellation (dodecagon-triangle tessellation)

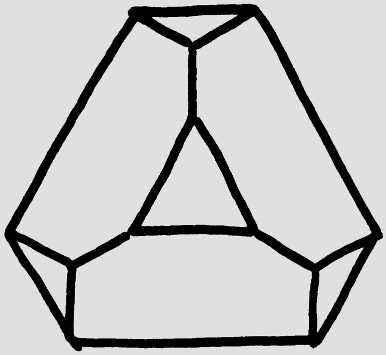


FIGURE 10-15 Vertex-truncated tetrahedron

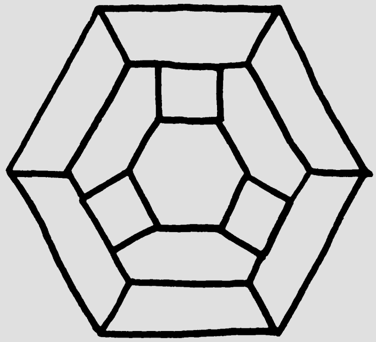


FIGURE 10-16 Vertex-truncated octahedron

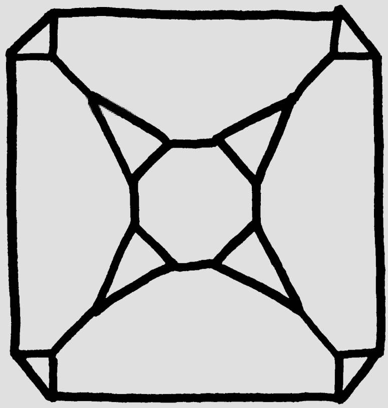


FIGURE 10-17 Vertex-truncated cube

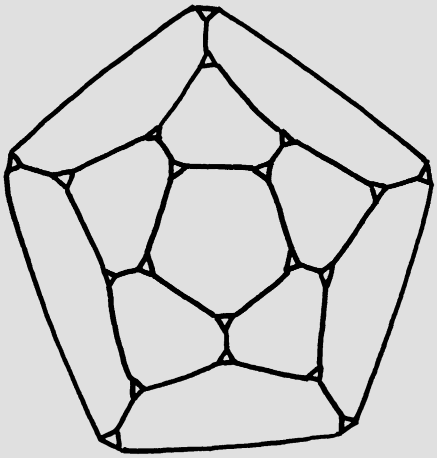


FIGURE 10-18 Vertex-truncated pentagonal dodecahedron

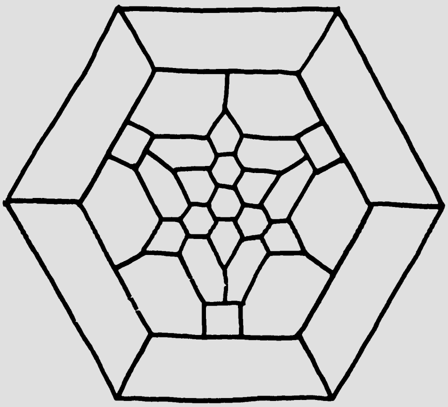


FIGURE 10-19 Vertex-truncated icosahedron

These expressions reduce, as before, to

$$V_{dt} = E \tag{10-7}$$

$$E_{dt} = 2E \tag{10-8}$$

$$F_{dt} = E + 2 \tag{10-9}$$

Table 10-3 accordingly summarizes the degenerate truncations of the regular solids; note that *dual structures* yield the same *degenerate*

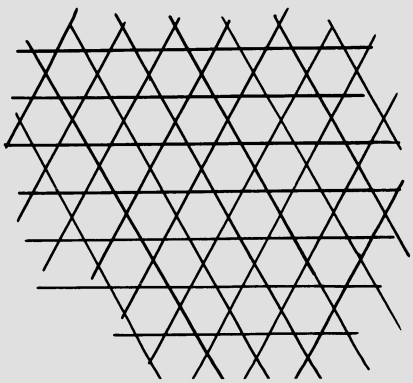


FIGURE 10-20
Degenerate truncation of triangular or hexagonal tessellations

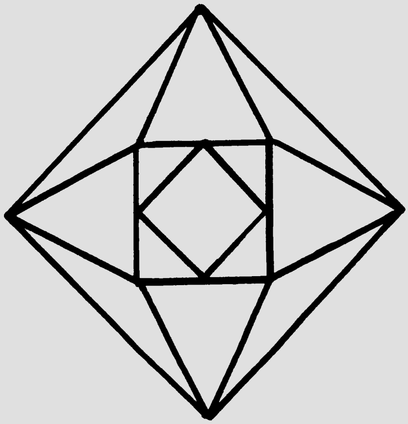


FIGURE 10-21 Cuboctahedron

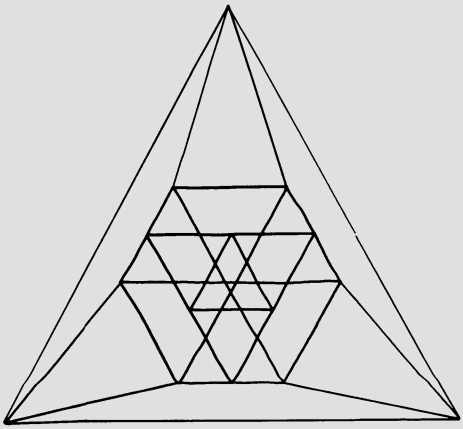


FIGURE 10-22 Icosidodecahedron

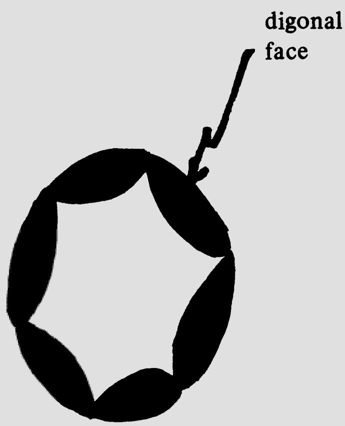


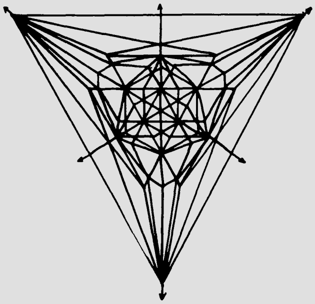
FIGURE 10-23 Degenerate truncation of a polygonal dihedron or digonal polyhedron

TABLE 10-3
Parameters of Solids Generated by Degenerate Truncation of Regular Solids

Regular solid	E	V_{dt}	E_{dt}	F_{dt}	*Specification of Faces*				*Name of truncated form*
					Corresponding to original faces		*Corresponding to original vertices*		
					Number	*Valency* (n)	*Number*	*Valency* (r)	
Tetrahedron	6	6	12	8	4	3	4	3	Octahedron
Octahedron	12	12	24	14	8	3	6	4	Cuboctahedron
Cube					6	4	8	3	
Icosahedron	30	30	60	32	20	3	12	5	Icosidodecahedron
Pentadodecahedron					12	5	20	3	

truncated solid. Note, further, that the result of degenerate truncation of the tetrahedron is *regular*—namely, the *octahedron.*

Degenerate truncation of the square tessellation yields a square tessellation in turn, having its vertices on the edges of the parent structure. Degenerate truncation of the triangular and hexagonal tessellation is illustrated in Fig. 10-20. The cuboctahedron is shown in Fig. 10-21, and the icosidodecahedron in Fig. 10-22. Degenerate truncation of a digonal *r*-hedron or an *r*-gonal dihedron yields a "scalloped lens" (Fig. 10-23).



11

Stellation and Truncation

A comparison of Figs. 10-1 and 10-2 demonstrates the following relationship between general truncation and general stellation:

1. Edge *truncation* replaces every edge by four edges and a new face; edge *stellation* replaces every edge by four edges and a new vertex.
2. Edge *truncation* replaces every r-valent vertex by a $2r$-valent face, every n-valent face by a $2n$-valent face. Edge *stellation* replaces every n-valent face by a $2n$-valent vertex, every r-valent vertex by a $2r$-valent vertex.
3. Edge *truncation* produces a structure having trivalent vertices only; edge *stellation* produces a structure having triangular faces only.

These observations lead to the conclusion that if, of a dual pair of of structures, we *truncate one* and *stellate* the other, or vice versa, we produce a new pair of duals.

In Fig. 10-2 we labeled edges that form part of the original edges α. Those joining new vertices to each other are labeled β, and those joining an original vertex to one replacing an original face, γ. In the absence of β-edges the original edges remain: faces are replaced by vertices having the same edge valency. This special stellation is called *face stellation.*

In the absence of both α- and β-edges every r-valent vertex is replaced by an r-valent face, and every n-valent face by an n-valent vertex, so that a *dual* results. When there are γ-edges, but no α-edges, this dual is stellated. In the absence of γ-edges only, special stellated structures result. (Cf. Chapter 12).

Finally, when γ-edges are the only ones present, all original edges are replaced by quadrilateral faces, and n-valent vertices replace the original faces. This instance is called *degenerate* stellation. These various stellations are illustrated in Fig. 11-1.

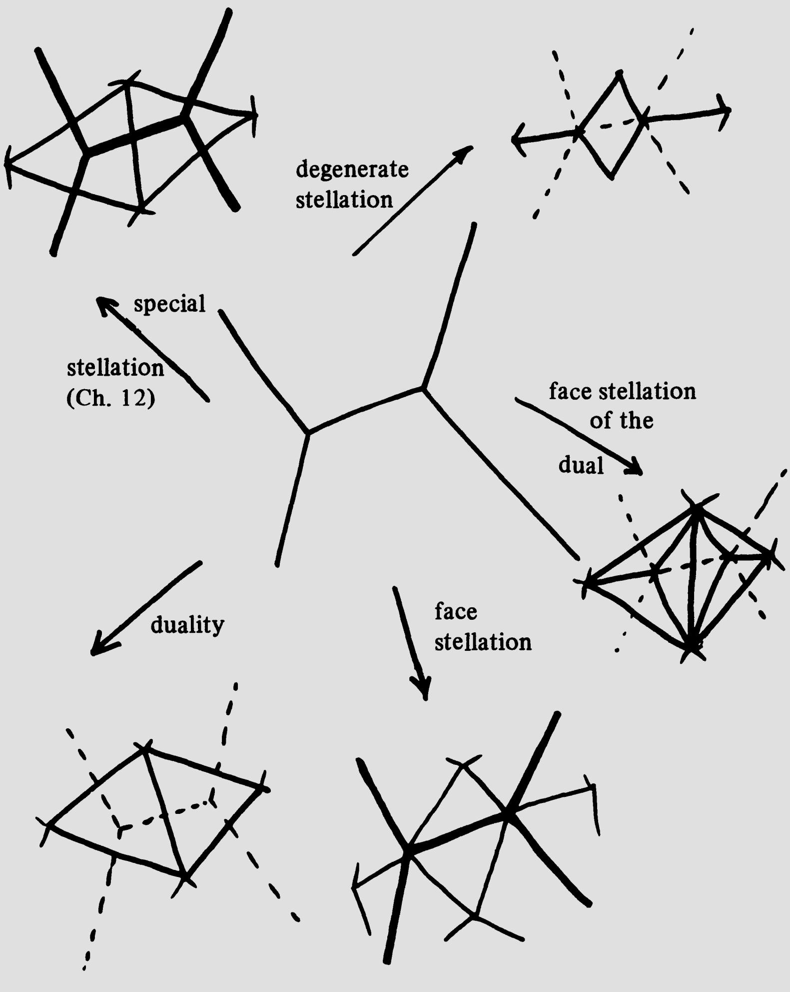


FIGURE 11-1 Special stellations

If we denote the parameters of the general edge-stellated structures by V_{Es}, E_{Es}, and F_{Es}, they are related to the parameters of the regular structures, V, E, and F, as follows:

$$V_{Es} = V + E + F; \quad E_{Es} = 2E + 2nF;$$
$$F_{Es} = 4E$$

From equation (4-6), $F = \frac{2}{n} E$; from equation (4-4), $V = \frac{2}{r} E$; and from equation (9-5), $\left(\frac{2}{r} + \frac{2}{n}\right)E = E + 2$.

$$\therefore\ V_{Es} = 2(E + 1) \tag{11-1}$$

$$E_{Es} = 6E \tag{11-2}$$

$$F_{Es} = 4E \tag{11-3}$$

Comparison with equations (10-1), (10-2), and (10-3) illustrates the duality of truncation and stellation.

Table 11-1 lists the results of stellating regular solids. The stellated structures have mutually equivalent triangular faces, and vertices of two different types. Figures 11-2, 11-3, and 11-4 are the *dual Schlegel diagrams* (cf. Chapter 8) of these stellated structures.

Figures 11-5 and 11-6 illustrate edge-stellated tessellations. A digonal polyhedron and a polygonal dihedron both yield bipyramids

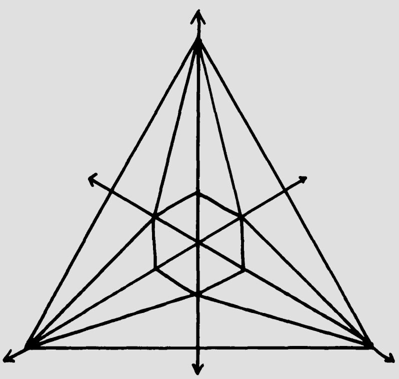


FIGURE 11-2 Edge-stellated tetrahedron (dual Schlegel diagram)

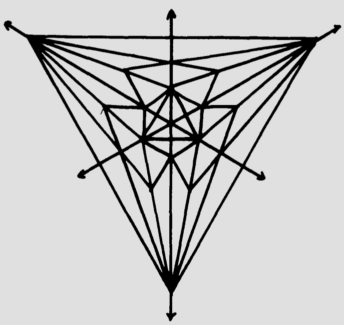


FIGURE 11-3 Edge-stellated octahedron or cube (dual Schlegel diagram)

TABLE 11-1
Edge Stellation of Regular Solids

Regular solid	*E*	V_{Es}	E_{Es}	F_{Es}	*Specification of Vertices*						*Name* of stellated solid*
					Corresponding to original vertices		*Corresponding to original edges*		*Corresponding to original faces*		
					Number	*Valency*	*Number*	*Valency*	*Number*	*Valency*	
Tetrahedron	6	14	36	24	4	6	6	4	4	6	Tetrakis hexahedron
Octahedron	12	26	72	48	6	8	12	4	8	6	Hexakis octahedron
Cube					8	6	12	4	6	8	
Icosahedron	30	62	180	120	12	10	30	4	20	6	Hexakis icosahedron
Pentagonal dodecahedron					20	6	30	4	12	10	

*The names used conform to R. Williams's nomenclature. Note that the hexakis octahedron could equally well be called octakis hexahedron, depending on whether the octahedron or the cube is regarded as the ancestor.

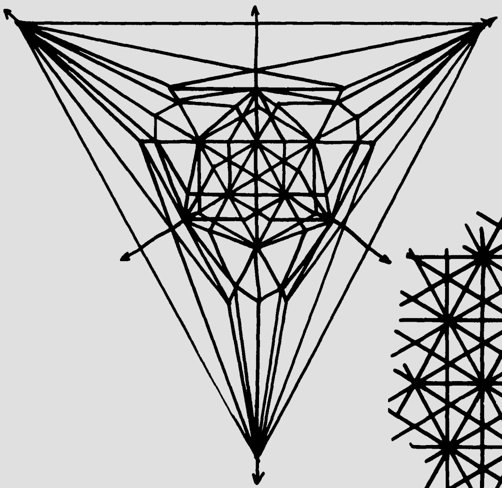


FIGURE 11-4 Edge-stellated icosahedron or pentagonal dodecahedron (dual Schlegel diagram)

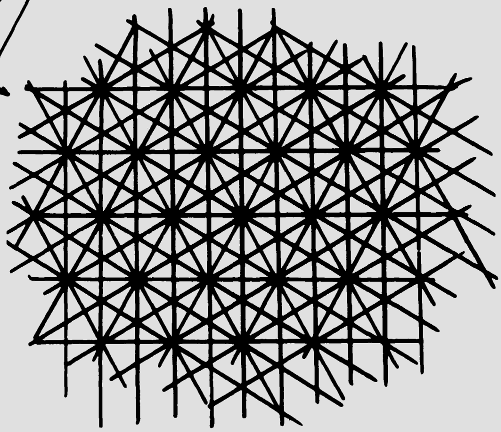


FIGURE 11-5 Edge-stellated triangular or hexagonal tessellation

(Fig. 11-7): a digonal r-hedron yields a $2r$-gonal bipyramid, while an n-gonal dihedron yields a $2n$-gonal bipyramid.

For face stellation the resulting parameters V_{Fs}, E_{Fs}, and F_{Fs} are

$$V_{Fs} = V + F; \quad E_{Fs} = E + nF; \quad F_{Fs} = 2E$$

Hence, using equations (4-4), (4-6), and (9-5), we obtain

$$V_{Fs} = E + 2 \tag{11-4}$$

$$E_{Fs} = 3E \tag{11-5}$$

$$F_{Fs} = 2E \tag{11-6}$$

The parameters of the face-stellated regular solids are listed in Table 11-2. Comparison of Table 11-2 with Table 10-2 confirms that when, of a dual pair of regular solids, either one is truncated, the other stellated, a new pair of duals results. These structures are illustrated in Figs. 11-8 through 11-12. The face-stellated tessellations are shown in Figs. 11-13 and 11-14; the face-stellated hexagonal tessellation is, in fact, a regular triangular tessellation.

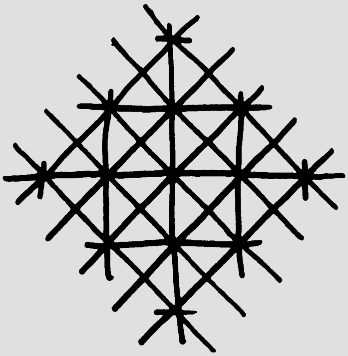


FIGURE 11-6 Edge-stellated quadrilateral tessellation

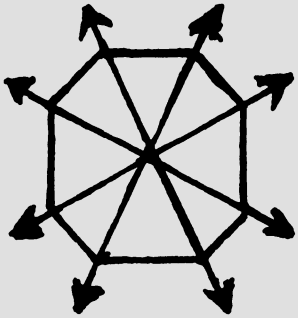


FIGURE 11-7 Bipyramid (dual Schlegel diagram) resulting from a digonal tetrahedron or quadrilateral dihedron by edge-stellation

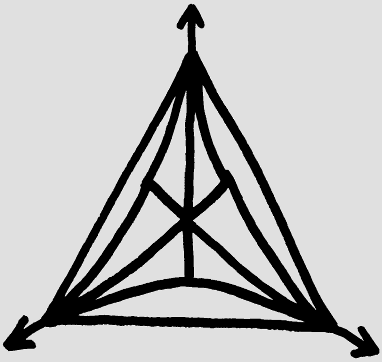


FIGURE 11-8 Face-stellated (tri-kis) tetrahedron (dual Schlegel diagram)

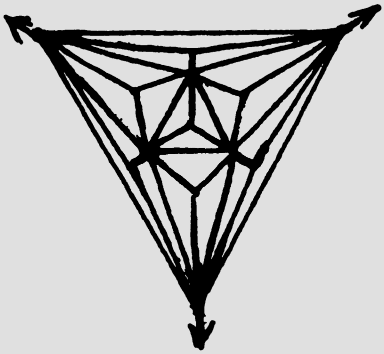


FIGURE 11-9 Face-stellated (trikis) octahedron (dual Schlegel diagram)

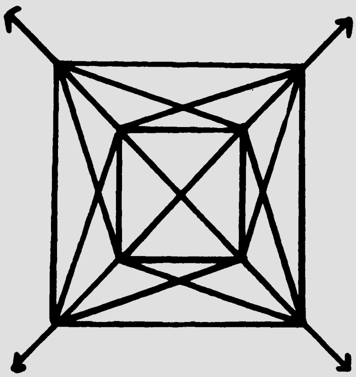


FIGURE 11-10
Face-stellated (tetrakis hexahedron) cube (dual Schlegel diagram)

TABLE 11-2
Face Stellation of Regular Solids

Solid	E	V_{Fs}	E_{Fs}	F_{Fs}	*Specification of Vertices*				*Name of stellated solid*
					Corresponding to original vertices		*Corresponding to original faces*		
					Number	*Valency* $(2r)$	*Number*	*Valency* (n)	
Tetrahedron	6	8	18	12	4	6	4	3	Triakis tetrahedron
Octahedron	12	14	36	24	6	8	8	3	Triakis octahedron
Cube	12	14	36	24	8	6	6	4	Tetrakis hexahedron
Icosahedron	30	32	90	60	12	10	20	3	Triakis icosahedron
Pentagonal dodecahedron	30	32	90	60	20	6	12	5	Pentakis dodecahedron

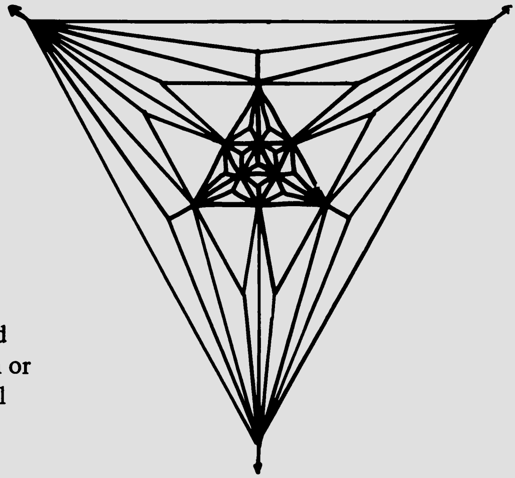


FIGURE 11-11 Face-stellated icosahedron (triakis icosahedron or hexecontahedron) (dual Schlegel diagram)

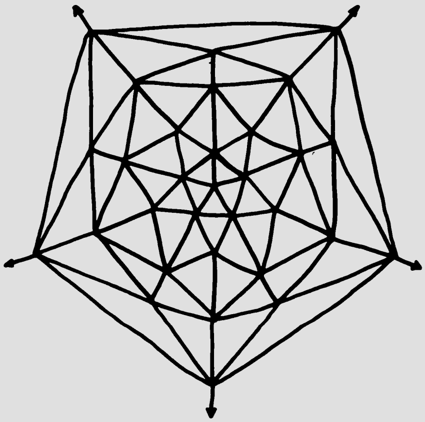


FIGURE 11-12 Face-stellated pentagonal dodecahedron (pentakis dodecahedron) (dual Schlegel diagram)

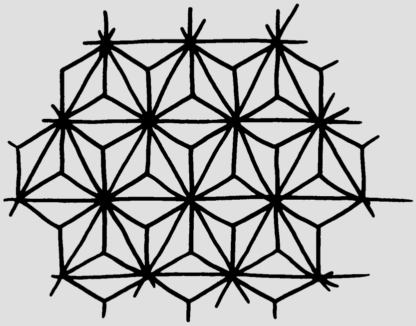


FIGURE 11-13 Face-stellated triangular tessellation

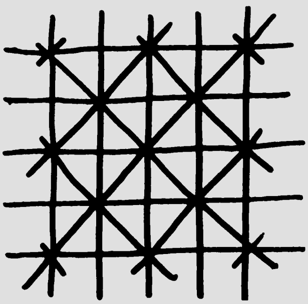


FIGURE 11-14 Face-stellated square tessellation

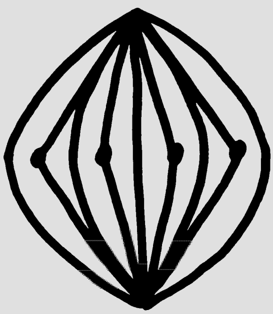


FIGURE 11-15 Face-stellated digonal polyhedron

Face stellation of an n-gonal dihedron yields an n-gonal bipyramid; face stellation of a digonal r-hedron yields a structure illustrated in Fig. 11-15, the dual of Fig. 10-12.

Degenerate stellation replaces every edge by a quadrilateral face, every face by a vertex, and preserves every original vertex. The parameters of degenerately stellated structures are

$$V_{ds} = E + 2 \tag{11-7}$$

$$E_{ds} = 2E \tag{11-8}$$

$$F_{ds} = E \tag{11-9}$$

The degenerately stellated regular solids are listed in Table 11-3; the name "rhombohedral triacontahedron" is entirely analogous to "rhombohedral dodecahedron," triaconta being Greek for 30. For the face-stellated forms the Greek suffix *kis* is often used; this indicates multiplication: *triakis* means "three times." A pentakis dodecahedron has five times twelve–i.e., sixty–faces, but is distinguished from other hexacontahedra (hexaconta = 60) by the identification of five faces for each original pentagonal face of the dodecahedron.

Comparison of Tables 10-3 and 11-3 confirms that octahedron and cube are indeed duals (truncation and stellation of the self-dual tetrahedron, respectively), and that the cuboctahedron and rhombohedral dodecahedron are also a dual pair (cf. Table 7-1). A

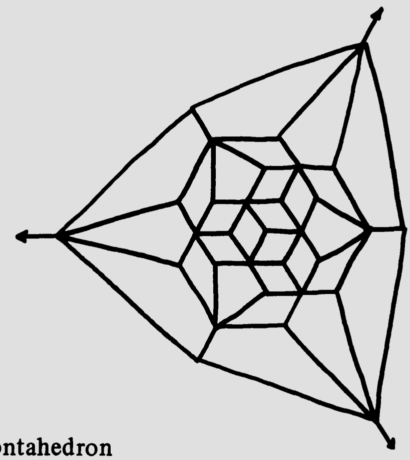


FIGURE 11-16
Rhombohedral triacontahedron

TABLE 11-3
Parameters of Solids Generated by Degenerate Stellation of Regular Solids

Regular Solids	E	V_{ds}	E_{ds}	F_{ds}	*Specification of Vertices* — *Corresponding to original vertices* — *Number*	*Specification of Vertices* — *Corresponding to original vertices* — *Valency* (r)	*Specification of Vertices* — *Corresponding to original faces* — *Number*	*Specification of Vertices* — *Corresponding to original faces* — *Valency* (n)	*Name of stellated form*
Tetrahedron	6	8	12	6	4	3	4	3	Hexahedron (cube)
Octahedron	12	14	24	12	6	4	8	3	Rhombohedral dodecahedron
Cube					8	3	6	4	
Icosahedron	30	32	60	30	12	5	20	3	Rhombohedral triacontahedron
Pentagonal dodecahedron					20	3	12	5	

Schlegel diagram of the rhombohedral dodecahedron is shown in Fig. 8-5; it is itself shown in Fig. 7-9. The rhombohedral triacontahedron, which is the dual of the icosidodecahedron, is shown in Fig. 11-16.

Degenerate stellation of an n-gonal dihedron yields a structure which has two polar, n-valent, vertices, which are joined by n quadrilateral faces; there are in total $(n + 2)$ vertices, $2n$ edges, and the n faces. The same structure is generated by degenerate stellation of a digonal polyhedron (Fig. 11-17). Degenerate stellation of the square tessellation yields a square tessellation; that of the triangular or hexagonal tessellation is shown in Fig. 11-18.

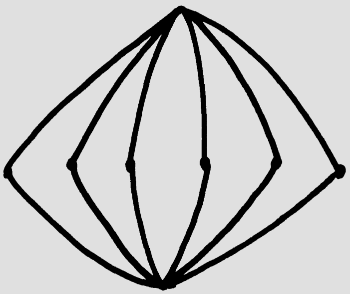


FIGURE 11-17 Degenerately stellated digonal polyhedron or polygonal dihedron

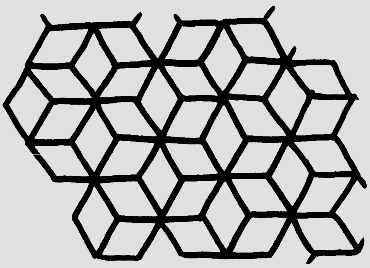


FIGURE 11-18 Degenerately stellated triangular or hexagonal tessellations

Table 11-4 summarizes the duality, truncation, and stellation relations between the regular and semiregular structures discussed. Radiating out from the center of this table are lines having at their opposite terminals dual pairs of structures: an edge-stellated cube is the dual of an edge-truncated octahedron, a rhombohedral dodecahedron the dual of a cuboctahedron, etc.

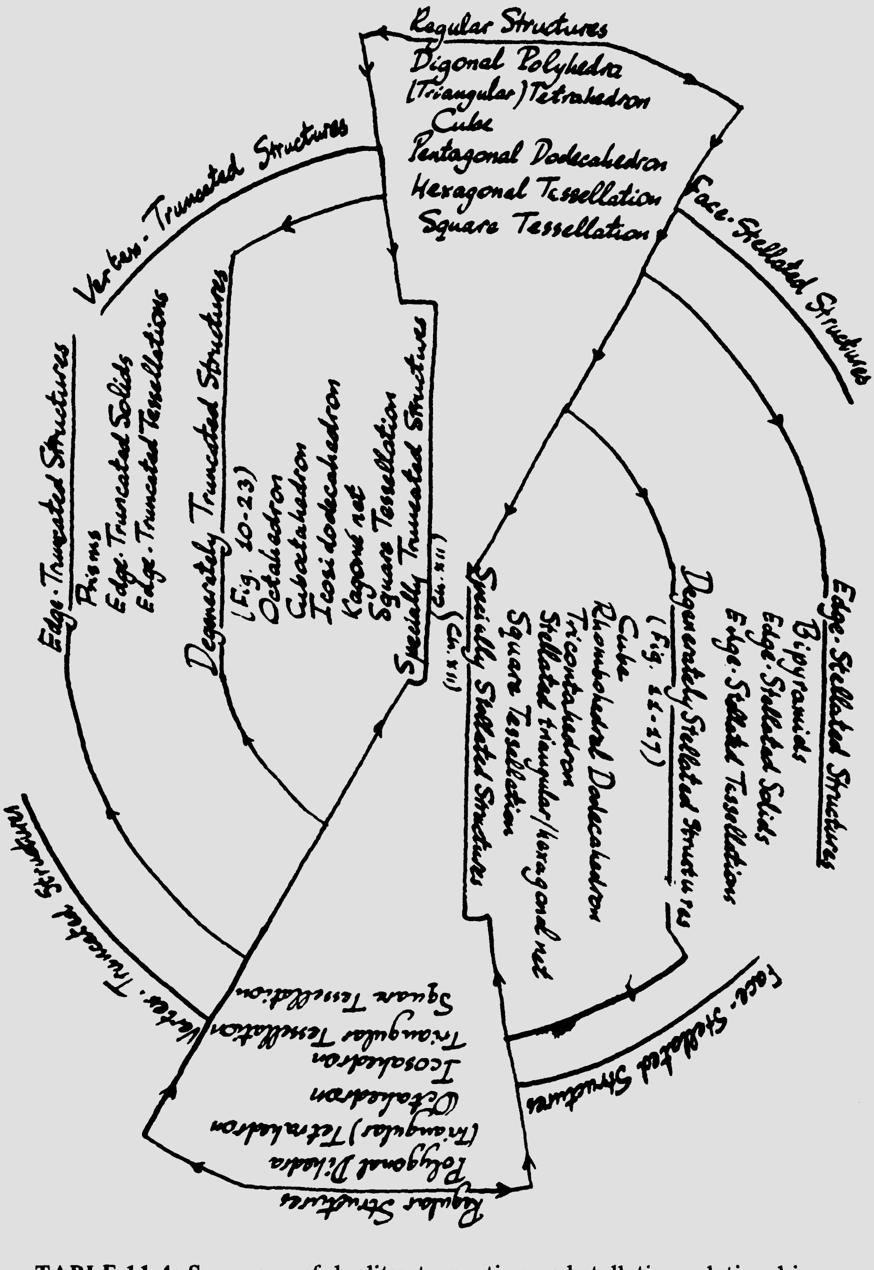
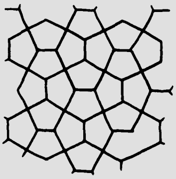
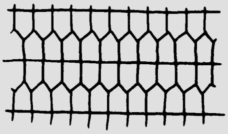


TABLE 11-4: Summary of duality, truncation and stellation relationships.





12

Exhaustive Enumeration of the Semiregular Two-dimensional Structures

In the preceding two chapters we generated semiregular structures by truncations and stellations of the regular structures found in Chapter 9. However, we shall find presently that we have by no means exhausted all possible semiregular structures in two dimensions. As we have found the means of determining the permitted connectivities to join the two-dimensional cells (polyhedra) into three-dimensional structures, it is important to ascertain that we are aware of all possible cells. To this purpose, we shall find all possible solutions of equation (9-5):

$$\frac{1}{\bar{r}} + \frac{1}{\bar{n}} = \frac{1}{2} + \frac{1}{l} \tag{9-5}$$

It will be noted that this equation is *symmetrical* in $\bar{r}$ and $\bar{n}$; that is to say, the equation is invariant to an interchange of $\bar{r}$ and $\bar{n}$. Therefore to every structure that satisfies equation (9-5) there corresponds its dual which also satisfies that equation. (Remember that, in order to preserve this symmetry in our structure of structures, a two-dimensional structure, in which *either* all faces *or* all vertices are equivalent, is called *semiregular.*)

Figure 12-1 demonstrates that we must further refine our definition of equivalence of faces, respectively vertices. In this structure all vertices are trivalent; from equation (9-5) it follows that, if it extends infinitely, $\bar{n} = 6$. Nevertheless, all vertices cannot be called equivalent: the combinations of polygons meeting at different vertices are not the same, even though these vertices are all trivalent. We shall require not only that two vertices, in order to be equivalent, have equal edge valencies, but that their environments be equivalent as well—i.e., that the structural elements adjacent to them have also the same valencies and occur in the same sequence, either clockwise or counterclockwise. The same requirement of equivalence applies to edges, faces, and cells. Formally, we can give a recursive definition of equivalence: two elements are equivalent if, and only if, they are adjacent to an equivalent set of elements arranged in identical or reverse order. This definition of *topological* equivalence is analogous to the definition of symmetry equivalence.[1]

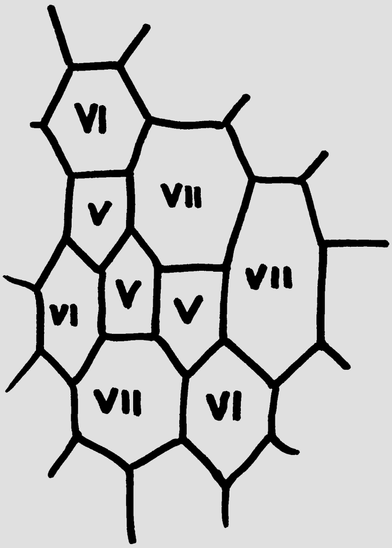


FIGURE 12-1
r=3, n=6. Roman numerals indicate value of n for each polygonal face. Locally, the number of edges per face may exceed or be below the value 6, but for an infinite net the average must be exactly 6.

Stover[2] has exhaustively enumerated regular and semiregular tessellations, assuming mutually *congruent* faces. Since in our own program we are not assuming congruencies, we shall here pursue the

[1] Cf. Arthur L. Loeb: *Color and Symmetry* (Wiley, New York, 1971), pp. 5 and 6.
[2] Donald Stover: *Mosaics* (Houghton Mifflin, Boston, 1966).

more general approach. This generalization is especially important in dealing with stellated structures, where we do not care whether the structure is convex or concave.

We shall explicitly discuss these structures having mutually equivalent faces, deriving from them their duals, which have mutually equivalent vertices.

Suppose that our structure has mutually equivalent faces, each of which has edge valency n. Suppose further that, of the n vertices of each face, n_1 have valency r_1, n_2 have valency $r_2, \ldots, n_i$ have valency r_i.

Since each r_i-valent vertex is shared by r_i faces,

$$\bar{r} = \frac{\sum_i (n_i/r_i)\, r_i}{\sum_i (n_i/r_i)} = \frac{n}{\sum_i (n_i/r_i)}$$

$$\therefore \frac{1}{\bar{r}} = \frac{1}{n} \sum_i \frac{n_i}{r_i} \tag{12-1}$$

When all vertices are equivalent, we deal with a *regular* structure, which has already been discussed. If there are *two* distinct vertex valencies, equation (9-5) gives

$$\frac{n_1}{r_1} + \frac{n_2}{r_2} = \left(\frac{1}{2} + \frac{1}{l}\right) n - 1 \tag{12-2}$$

$$n_1 + n_2 = n$$

Hence

$$n_1 = r_1 \frac{\left(\frac{l + 2}{2l}\, n - 1\right) r_2 - n}{r_2 - r_1} \tag{12-3}$$

$$n_2 = r_2 \frac{n - \left(\frac{l + 2}{2l}\, n - 1\right) r_1}{r_2 - r_1} \tag{12-4}$$

[These equations are analogous to equations (6-4) and (6-5).]

We shall assume that $r_2 > r_1$: this is *not* a limiting assumption, but simply assigns subscripts. If $r_1 = 2$, all *other* vertices have to be mutually equivalent; the resulting structures would be regular ones with additional divalent vertices on each of the edges. Such structures would be trivially semiregular, and will be ignored here. Hence $r_1 \geqslant 3$, and $\bar{r} > 3$. From equation (9-5),

$$n < \frac{6l}{l + 6}$$

For an infinitely extended tessellation $l = \infty$; then $n = 6$ only if $\bar{r} = 3$, a pair of values corresponding to the regular hexagonal tessellation of the plane. Hence for the semiregular tessellations and polyhedra $n < 6$. The last inequality may also be usefully inverted:

$$l > \frac{6n}{6 - n}$$

Moreover, for semiregular structures $n_1 > 0$ and $n_2 > 0$; from equations (12-3) and (12-4):

$$r_1 < \frac{n}{\left(\frac{l + 2}{2l} n - 1\right)} < r_2$$

Using these inequalities, we can systematically find all possible combinations of r_1 and r_2 for each of the possible values $n = 2, 3, 4,$ and 5. It is not possible to have a digonal structure in which the two vertices of each face have different valencies; hence $n_1 \neq 2$.

First, there are two different combinations of possible values of n_1 and n_2 ($n_1 = 1, n_2 = 2$; and $n_1 = 2, n_2 = 1$) when $n = 3$. Figure 12-2 shows triangular faces of a semiregular structure whose vertex valencies are r_a and r_b. If *two* vertices of every face have valencies r_a, and the third vertex is r_b-valent, then every r_a-valent vertex is surrounded by a polygon whose vertices are alternatively r_a-valent and r_b-valent. Therefore r_a must necessarily be even, and hence, when $n_1 = 1, r_2$ is even, whereas when $n_2 = 1, r_1$ is even. The possible combinations of parameters are listed in Table 12-1; each corresponds to a semiregular, face-stellated structure discussed in the previous pair of chapters. Especially interesting is the relation $l = 3r_2$ for the case $n_1 = 2, n = 1, r_1 = 4$, which in one sweep generates all bipyramids, the trigonal bipyramid in particular corresponding to the

TABLE 12-1
Possible Values for n_1, n_2, r_1, r_2, and l for Triangulated Structures Having Two Different Vertex Valencies

n_1	n_2	r_1	r_2	l	*Structure*
1	2	3	4	9	Triangular bipyramid (face-stellated triangle)
			6	18	Triakis (face-stellated) tetrahedron
			8	36	Triakis (face-stellated) octahedron
			10	90	Triakis (face-stellated) icosahedron (Fig. 11-11)
			12	∞	Face-stellated triangular tessellation (Fig. 11-13)
		4	6	36	Tetrakis hexahedron (face-stellated cube) (Fig. 11-10)
			8	∞	Face-stellated square tessellation (Fig. 11-14)
		5	6	90	Pentakis (face-stellated) dodecahedron (Fig. 11-12)
2	1	4	$l = 3r_2$		Bipyramids (face-stellated dihedral polygons)

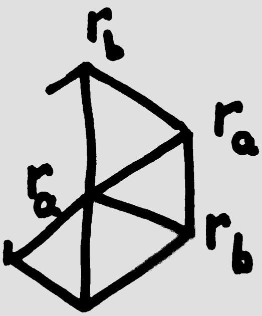


FIGURE 12-2 Structure having triangular faces and two possible vertex valencies

first entry in Table 12-1. (Since it, of all bipyramids, has a polar valency less than that of the equatorial vertices, it made its appearance separately. The *square* bipyramid is a *regular* structure: the octahedron.)

Thus we exhaust the structures having mutually equivalent triangular faces and *two* kinds of vertices. Structures having only triangular faces are called *deltahedra*. An enumeration of all possible *equilateral* deltahedra has been given by Freudenthal and van der Waerden.[3,4] This enumeration differs from the present one in that faces do not need to be mutually *equivalent,* but we, on the other hand, do not require the faces to be *equilateral.* It should be noted that the values tabulated in Table 12-1 were found by solving equation (12-2) without the restriction of mutual equivalence of the faces, but the interpretation as stellated regular structures does implicitly use that assumption. We shall return to the deltahedra when we consider more than two different valencies of vertices.

For quadrilateral faces $n = 4$; there are three possible combinations of n_1 and n_2: (1, 3), (2, 2), and (3,1). The permitted solutions of equation (12-2) for each of these combinations are listed in Table 12-2. The first structure listed is a new one, having twenty-four faces (it is an icositetrahedron), which could be generated *either* from the cube *or* from the octahedron by replacing each face by a vertex, and joining the new vertices to each other across the intervening edges. This is the special type of stellation corresponding to $\gamma = 0$ in Fig. 10-2. It could also be considered as the superposition of two dual structures (cube and octahedron), causing new vertices to appear where their edges intersect. Such structures actually have

TABLE 12-2
Possible Values for n_1, n_2, r_1, r_2, and l
for Quadrilateral Structures Having Two Different Vertex Types

n_1	n_2	r_1	r_2	l	*Corresponding structure*
1	3	3	4	48	Trapezoidal icositetrahedron (Figs. 12-3 and 12-4)
2	2	3	4	24	Rhombohedral dodecahedron (Figs. 7-9 and 8-5)
			5	60	Degenerate stellated icosahedron/pentagonal dodecahedron (Fig. 11-16) (rhombohedral triacontahedron)
			6	∞	Degenerate stellated triangular/hexagonal tessellation
3	1	3	$r_2 = \frac{1}{4} l$		

[3] H. Freudenthal and B. L. van der Waerden: "Over een bewering van Euclides" (Simon Stevin, **25**, 115-121).

[4] M. Walter: *On Constructing Deltahedra,* to be published.

three different sets of vertices: those of the original pair of duals, as well as the new quadrivalent ones at the edge intersections. The fact that one of this type of structure turns up when *two* vertex valencies are considered is caused by the tetravalency of the original octahedron vertices. Again, this is an illustration of the fact that nothing in the solution of equation (12-2) uses the assumption of equivalency of equi-valent faces or vertices (note that equivalent ≠ equi-valent!). Williams[5] points out that the icositetrahedron of Fig. 12-3 may be transformed into another by slicing it into two halves, twisting these halves with respect to each other before reattaching them. Both icositetrahedral forms are shown as dual Schlegel diagrams in Fig. 12-4. It is observed that the first form has *six* tetravalent vertices that are joined *only* to other tetravalent vertices, the remaining twelve being joined to two trivalent and two tetravalent vertices. In

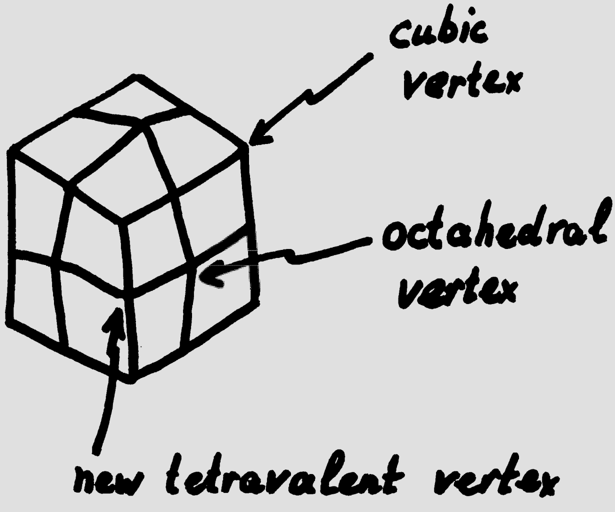


FIGURE 12-3 Icositetrahedron

the second structure two tetravalent vertices are joined to four other tetravalent vertices each, the remaining sixteen having a single trivalent and three tetravalent vertices.

For $n_1 = n_2 = 2$ we find the results of degenerate stellation of the respective pairs octahedron/cube (i.e., the rhombohedral dodecahedron), icosahedron/dodecahedron (i.e., the rhombohedral triacontahedron), and the triangular/hexagonal tessellations.

[5] Robert Williams: *Natural Structure* (Eudaemon Press, Moorpark, California, 1972).

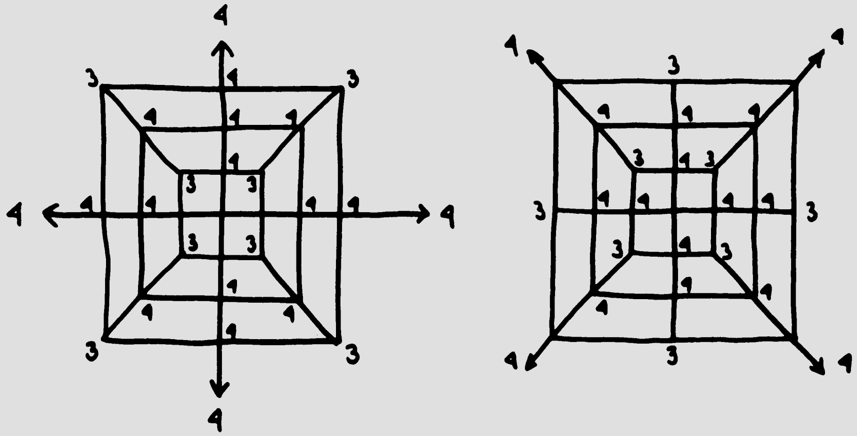
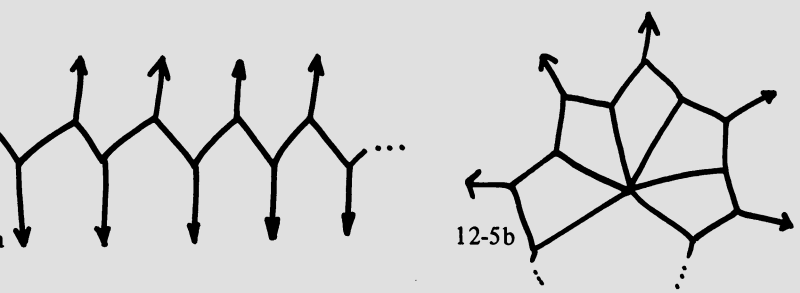


FIGURE 12-4 Dual Schlegel diagrams of two icositetrahedral forms

For $n_1 = 3$, $n_2 = 1$ the sole solution is $r_2 = \frac{1}{4}l$, which corresponds to a significant family of structures not yet encountered here, called trapezohedra. They are illustrated in Fig. 12-5 in a Mercator projection as well as in a dual Schlegel diagram. They can be described as having a staggered polygon as equator, having $2r_2$ edges. Alternate vertices of the equatorial polygon are joined to each of the two r_2-valent polar vertices; r_2 may have any finite or infinite integral value; l must be an integral multiple of 4.

The possible solutions of equation (12-2) for $n = 5$ are listed in Table 12-3. Two of the corresponding structures are tessellations. The one having both trivalent and tetravalent vertices may appear in two forms, depending on whether two tetravalent vertices are adjacent to each other, or separated by a trivalent vertex. Both forms are illustrated in Fig. 12-6.



12-5a 12-5b

FIGURE 12-5 Trapezohedron (a) Mercator projection (b) Dual Schlegel diagram

TABLE 12-3
Possible Values for n_1, n_2, r_1, r_2, and l for Pentagonal Structures Having Two Different Vertex Types

n_1	n_2	r_1	r_2	l	*Corresponding structure*
3	2	3	4	∞	Pentagonal semiregular tessellation (Fig. 12-6)
4	1	3	4	60	Pentagonal icositetrahedron (Fig. 12-8)
4	1	3	5	150	Pentagonal hexecontahedron (Fig. 12-9)
4	1	3	6	∞	Skew pentagonal semiregular tessellation (Fig. 12-7)

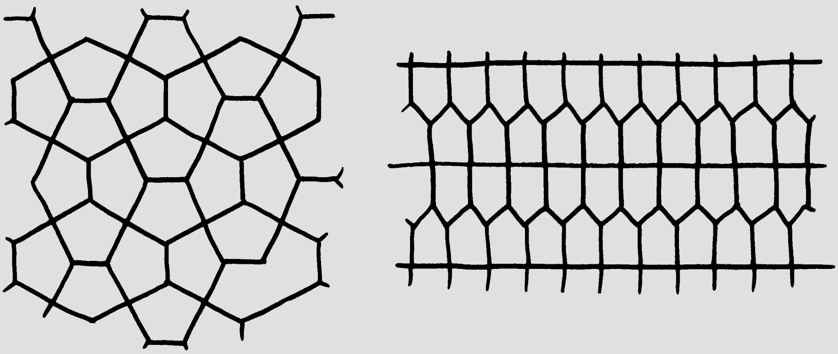


FIGURE 12-6 Two tessallations having $n_1 = 3$, $n_2 = 2$, $r_1 = 3$, $r_2 = 4$

The other tessellation is shown in Fig. 12-7. It is a new structure for us, and it has the following peculiarity. The trivalent vertices are of two different types, namely those joined to other trivalent ones only, and those that have one hexavalent and two trivalent neighbors. If we go around a pentagonal face in clockwise direction, then the valencies of its vertices are 63333. If we denote the single trivalent vertex joined to other trivalent vertices only by a prime, we have two possibilities: 633′33 and 6333′3. The latter combination corresponds to Fig. 12-7, the former to a mirror image of Fig. 12-7. This structure therefore exists in two *enantiomorphic* manifestations: we call such a structure *skew*.

The remaining two sets of parameters correspond to skew derivatives, respectively, of the cube (octahedron) and of the pentagonal

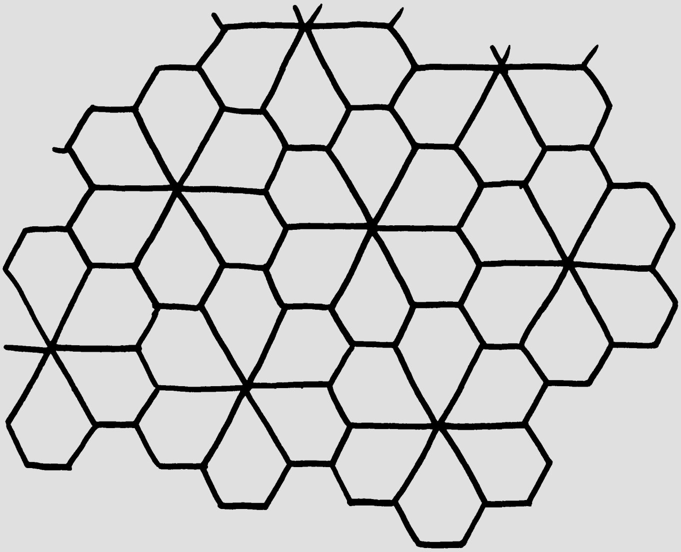


FIGURE 12-7 Tessellation having $n_1=4, n_2=1, r_1=3, r_2=6$

dodecahedron (icosahedron). Since they have, respectively, twenty-four and sixty faces, they are called icositetrahedron and hexacontahedron; they are illustrated in Figs. 12-8 and 12-9. Both have, in each pentagonal face, a single trivalent vertex connected to trivalent vertices only, from which they derive their skew nature. They both exist in enantiomorphic manifestations.

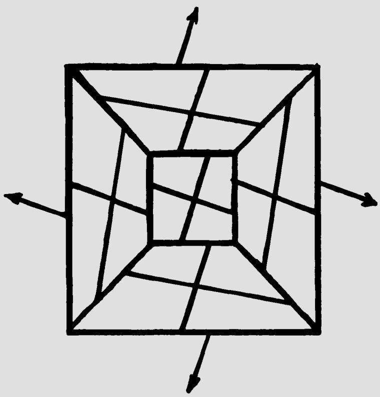


FIGURE 12-8
Pentagonal icositetrahedron

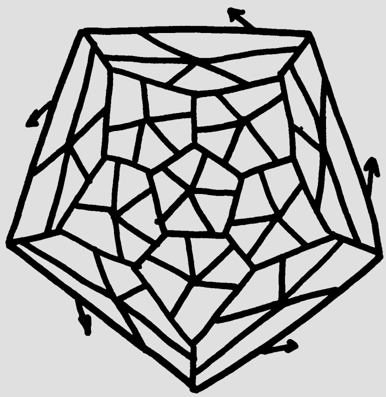


FIGURE 12-9
Pentagonal hexacontahedron

These structures exhaust the possibilities for exactly two different vertex valencies. For *three* different vertex valencies equation (9-5) becomes

$$\frac{n_1}{r_1} + \frac{n_2}{r_2} + \frac{n_3}{r_3} = \left(\frac{1}{2} + \frac{1}{l}\right)n - 1 \tag{12-5}$$

For triangular faces $n_1 = n_2 = n_3 = 1$. If we consider a triangular face whose vertex valencies are r_a, r_b, and r_c, then every r_a-valent vertex is surrounded by r_a triangles which constitute a stellated r_a-gon whose vertices have, alternately, the valencies r_b and r_c. Therefore r_a, and by an analogous argument r_b and r_c, must be even. Thus r_1, r_2, and r_3 in equation (12-5) must be even when $n = 3$. For the case $r_1 = 4$, we find from equation (12-5)

$$\frac{1}{r_3} = \frac{1}{4} + \frac{3}{l} - \frac{1}{r_2} < \frac{1}{r_2}$$

$$\therefore\ r_2 < \frac{8l}{l + 12} \leqslant 8$$

Thus, if $r_1 = 4$, r_2 could be either 6 or 8. However, if $l = \infty$ and $r_2 = 8$, then $r_3 = 8 = r_2$, which violates $r_2 \neq r_3$, and hence reduces this case to one already considered (*two* different vertex valencies instead of *three*). Therefore we are reduced to just three possible triangular structures, listed in Table 12-4, which are all edge-stellated forms.

TABLE 12-4
Triangulated Structures Having Three Different Vertex Valencies

r_1	r_2	r_3	l	*Structure*
4	6	8	72	Edge-stellated cube/octahedron (hexakis octahedron or octakis hexahedron) (Fig. 11-3)
		10	180	Edge-stellated dodecahedron/icosahedron (hexakis icosahedron or decakis dodecahedron) (Fig. 11-4)
		12	∞	Edge-stellated hexagonal/triangular tessellation (Fig. 11-5)

For quadrilaterally faced structures equation (12-5) becomes

$$\frac{n_1}{r_1} + \frac{n_2}{r_2} + \frac{n_3}{r_3} = 1 + \frac{4}{l}$$

The inequality $r_1 < r_2 < r_3$ then becomes, by solving for r_3 in terms of r_1 and r_2,

$$r_1 < r_2 < r_1 \frac{n_3 + n_2}{\left(1 + \frac{4}{l}\right) r_1 - n_1}$$

The extreme members of these inequalities yield:

$$\left(1 + \frac{4}{l}\right) r_1 < n_1 + n_2 + n_3 = 4;$$

hence $r_1 = 3$, and

$$3 < r_2 < \frac{3(4 - n_1)l}{3(l + 4) - n_1 l}$$

The extreme values of this inequality yield $l > 12$; the upper limit of the right-hand side of this inequality is $3(4 - n_1)/(3 - n_1)$. The value of n_1 is *either* 1 *or* 2; when $n_1 = 1$, r_2 is limited to 4, whereas when $n_1 = 2$, r_2 may equal either 4 or 5.

For $n_1 = 1$:

$$\frac{n_2}{4} + \frac{n_3}{r_3} = \frac{2}{3} + \frac{4}{l}$$

There are now two possibilities: $n_2 = 1, n_3 = 2$, or $n_2 = 2, n_3 = 1$. The former combination limits r_3 to values less than 5; since it *must* exceed r_2, this is not possible. Therefore, if $n_1 = 1$, then $n_2 = 2$, $n_3 = 1$, and hence

$$r_3 = \frac{6l}{l + 24}, \qquad l = \frac{24 r_3}{6 - r_3}$$

These expressions limit r_3 and l to two possibilities: $r_3 = 5, l = 120$, and $r_3 = 6, l = \infty$.

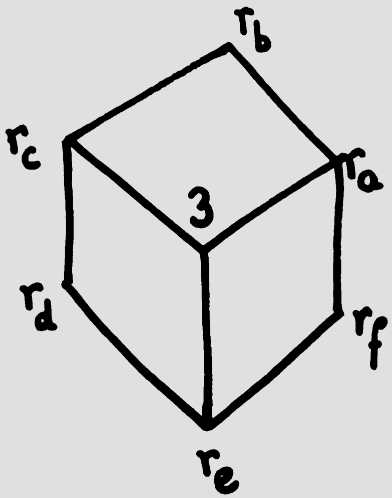


FIGURE 12-10 Three mutually equivalent faces joined at a trivalent vertex

Figure 12-10 shows three mutually equivalent quadrilateral faces meeting at a trivalent vertex; the equivalence of the faces requires that $r_a = r_c = r_e$, and $r_b = r_d = r_f$. The first of these two sets of equalities implies that $r_b \neq 3$, for if $r_b = 3$, then there are only *two* different vertex valencies on each face. However, if $r_b \neq 3$, and $n_1 = 2$, then $r_a = r_c = 3$; but this again prevents the possibility of three separate valencies. Therefore $n_1 \neq 2$, since its consequences are all inadmissible. We therefore conclude that there are only *two* possible quadrilaterally faced semiregular structures having three different vertex valencies, as listed in Table 12-5. Both of these are an analogous to the superimposed duals illustrated in Fig. 12-3 generating the icositetrahedron (deriving from Fig. 10-2 by letting γ vanish). We had already noted there that its quadrilateral faces have *three* different *types* of vertices, even though two of these types both have

TABLE 12-5
The Two Quadrilaterally Faced Semiregular Structures Having Three Distinct Vertex Valencies

n_1	n_2	n_3	r_1	r_2	r_3	l	*Structure*
1	2	1	3	4	5	120	Trapezoidal hexecontahedron
					6	∞	Superimposed hexagonal/triangular tessellation

valency 4. The number of faces of this icositetrahedron equals the smallest number into which the numbers of faces of its component cube (six faces) and octahedron (eight faces) are divisible. It is not surprising, therefore, that the icosahedron (twenty faces) and dodecahedron (twelve faces) would generate a quadrilateral hexacontahedron (sixty faces). Both structures listed in Table 12-5 are illustrated, one in Fig. 12-11, the other in Fig. 12-12.

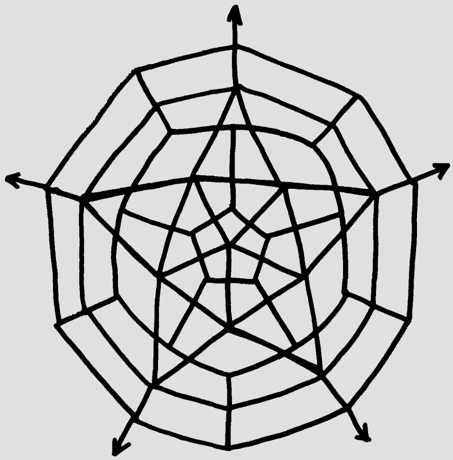


FIGURE 12-11 Trapezoidal hexecontahedron

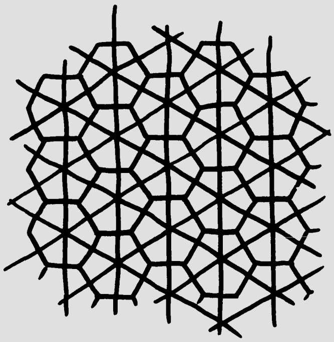


FIGURE 12-12 Tessellation resulting from the superposition of a triangular and hexagonal net

It is interesting to consider the results of superimposing the regular duals analogously. The digonal polyhedra and polygonal dihedra generate bipyramids having a divalent vertex on each equatorial edge; since we explicitly declared such structures as trivial derivatives of other structures, they were not generated separately in this chapter. Superposition of two tetrahedra produces a structure having *eight* trivalent vertices and *six* quadrivalent vertices–i.e., our old friend, the rhombohedral dodecahedron, already generated. Two square tessellations superimpose to yield a square tessellation.

For pentagonal structures having three different vertex valencies equation (12-5) becomes:

$$\frac{n_1}{r_1} + \frac{n_2}{r_2} + \frac{n_3}{r_3} = \frac{3}{2} + \frac{5}{l}$$

The minimal value of the right-hand side of this equation is 3/2. The maximal value of the left-hand side occurs when $n_1 = 3, n_2 = n_3 = 1$,

$r_1 = 3, r_2 = 4, r_3 = 5$; the left side then equals 29/20, just short of the minimal value of the right-hand side. Therefore it is impossible to have pentagonal structures having more than two different vertices: terms n_4/r_4 and n_5/r_5 would further diminish the left-hand side of equation (12-5).

With *four* different valencies one might have quadrilateral faces; equation (9-5) then becomes

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4} = 1 + \frac{4}{l}$$

The maximum value of the left-hand side of this equation is 57/60, just short of the minimal value of 1 for the right-hand side. We see, therefore, that we have thus exhausted all possible semiregular structures having mutually equivalent faces.

Summarizing, we have found in this exhaustive enumeration of semiregular structures:

1. All edge-stellated, face-stellated, and degenerate stellated regular structures found in Chapter 11.
2. The trapezohedra (Fig. 12-5).
3. The dual superpositions (Figs. 12-3, 12-4, 12-11, and 12-12), corresponding to the special stellations for which γ vanishes in Fig. 10-2.
4. A pair of pentagonal tessellations having $r_1 = 3, r_2 = 4$ (Fig. 12-6).
5. The skew structures (Figs. 12-7, 12-8, 12-9).

At the beginning of this chapter we said that, once the structures having mutually equivalent *faces* are found, their *duals* follow, to complete the enumeration of *all* semiregular structures. In Chapter 10 we found the duals of the stellated structures—namely, the truncated ones. The duals of trapezohedra are antiprisms, structures having two polygonal faces joined by triangular faces (Fig. 12-13); all vertices are tetravalent.

The specially stellated forms having $\gamma = 0$ were interpreted as superimposed dual pairs. The duals of these forms correspond to special *truncations,* having $c = 0$ in Fig. 10-1. The resulting structures have faces corresponding to the faces of both generating dual pairs, as well as quadrilateral faces corresponding to the edges of the generating pair. The duals of the icositetrahedra of Figs. 12-3 and

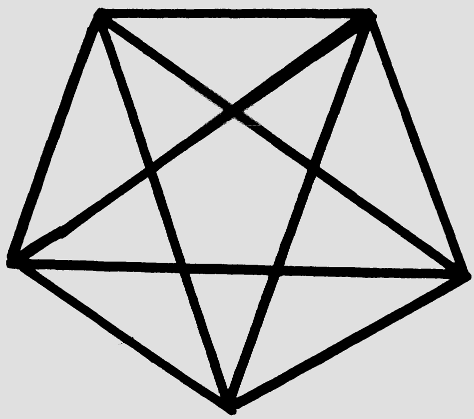
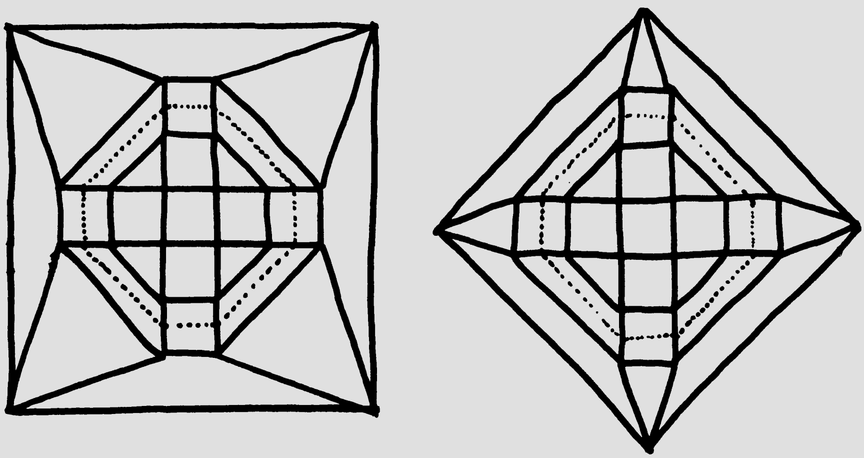


FIGURE 12-13 Pentagonal antiprism

12-4 have eight triangular faces, corresponding to the eight octahedral faces, six quadrilateral faces, corresponding to the faces of the cube, and twelve additional quadrilateral faces corresponding to the cube/octahedron edges. The Schlegel diagrams of both forms are shown in Fig. 12-14; the dotted line denotes the plane along which the two halves would be severed prior to twisting the two halves in order to transform one structure into the other. This form is known as the small rhombicuboctahedron.



(a) Small rhombicubocatahedron (b) Twist small rhombicuboctahedron

FIGURE 12-14 Dual of icositetrahedral structure

Analogously, the trapezoidal hexecontahedron will have a dual having twenty (icosahedral) triangular faces, twelve (dodecahedral) pentagonal faces, and thirty quadrilateral faces corresponding to the edges of the generating duals; there are *sixty* quadrivalent vertices (Fig. 12-15). Similarly, the tessellation of Fig. 12-12 has a dual, shown in Fig. 12-16, and the two tessellations of Fig. 12-6 have as duals the tessellations shown in Fig. 12-17.

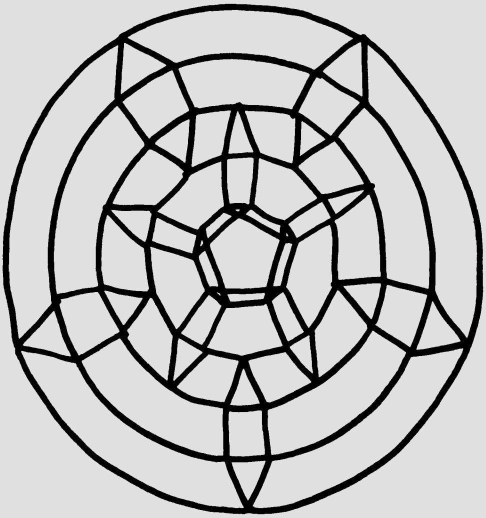


FIGURE 12-15 Dual of the trapezoidal hexecontahedron: the small rhombicosidodecahedron

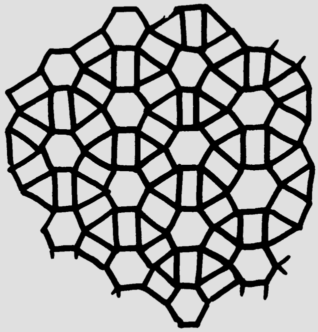


FIGURE 12-16 Tessellation having quadrivalent vertices, triangular, quadrilateral, and hexagonal faces

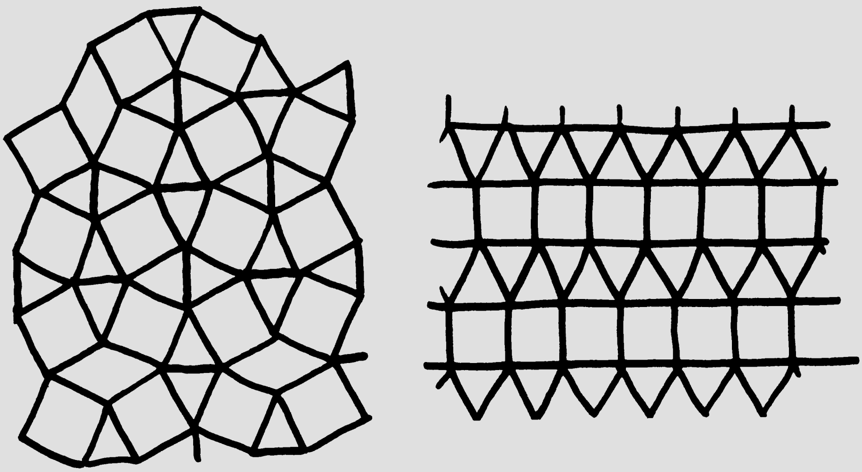


FIGURE 12-17 Tessellations having pentavalent vertices, quadrilateral and triangular faces

The skew structures of Figs. 12-7, 12-8, and 12-9 also have duals, illustrated in Figs. 12-18, 12-19, and 12-20, respectively. These duals are called *snub* polyhedra; they exist in enantiomorphic manifestations.

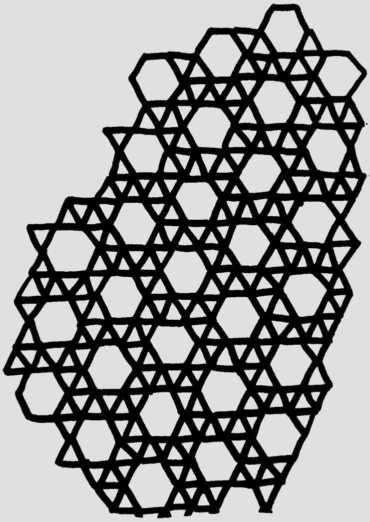


FIGURE 12-18 Snub tessellation of triangles and hexagons, having pentavalent vertices

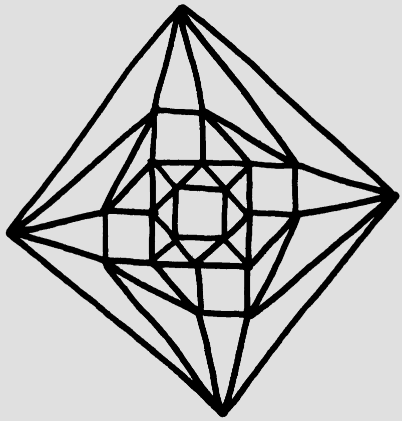


FIGURE 12-19 Snub cube

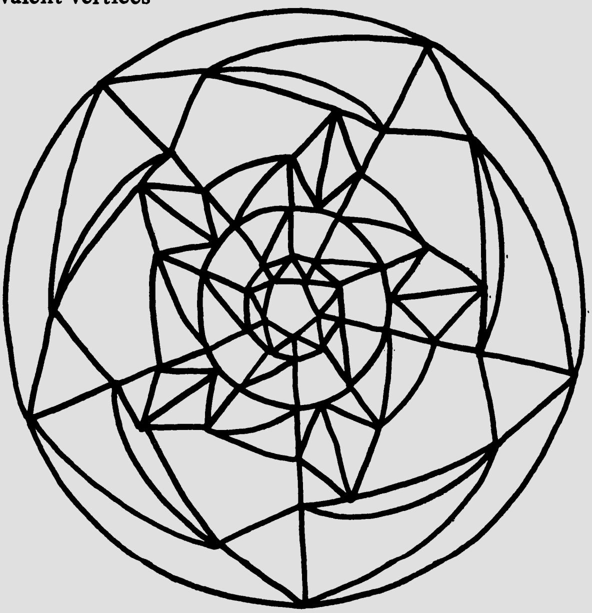
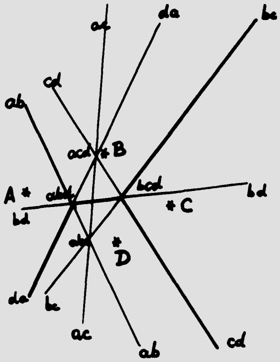


FIGURE 12-20 Snub dodecahedron

Thus we conclude the exhaustive enumeration and classification of semiregular structures; we have shown that, except for the *skew* forms and their duals, the *snub* forms, and for the single special pentagonal tessellation and its dual, they can be systematically related to the regular structures by the general truncations demonstrated in Figs. 10-1 and 10-2.



13

Dirichlet Domains in a Plane[1]

With the enumeration of all regular and semiregular cells and the rules for combining them into three-dimensional structures, we have completed the discussion of connectivities, and are ready to take actual distances into account.

Consider an assembly of discrete points—for instance, all the churches of a given denomination, all elementary schools, or all subway stations in a city. It might be useful to subdivide the city into parishes, school districts, or subway districts in such a way that *everyone* living in a given district lives closer to the church, school, or subway station in his own district than to any other church, school, or subway station. The parish or district so defined is called a *Dirichlet Domain;* the church, school, or subway station is called its *center.*

Generally, a Dirichlet Domain of a particular member of a set of discrete points is defined as that region which contains all locations closer to that particular point than to all other points of the set. We shall first consider Dirichlet Domains for points not regularly spaced, and then apply the discussion to regularly spaced arrays of points.

[1] Chapters 13 and 14 are adapted from a contribution to a Festschrift dedicated to Rudolf Arnheim on his seventieth birthday, 1974.

The construction of Dirichlet Domains depends on finding the locus of all points equidistant from two given points. This locus is the perpendicular bisector of the line joining the points (Fig. 13-1). This bisector divides the plane into two regions; within either region every location is closer to the particular one of the pair of points which that region contains than to the other point.

In Fig. 13-2 there are three points, A, B, and C; the line segments $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$ must all be perpendicularly bisected in order to divide space into the appropriate three domains. The bisector of $\overline{AB}$ is labeled ab, that of $\overline{BC}$ is labeled bc, that of $\overline{CA}$ is labeled ca. All points on ab are equidistant from A and B, those on bc are equidistant from B and C. The point of intersection of ab and bc is therefore equidistant from A, B, and C, hence lies on ca as well. This point, the intersection of all *three* bisectors, is called O; it is the center of the circle passing through A, B, and C. Since A and B are equidistant from ab, the angles which the lines OA and OB make with ab are equal; we shall call this angle γ. The angles α and β in Fig. 13-2 are defined analogously; $\alpha + \beta + \gamma = 180°$.

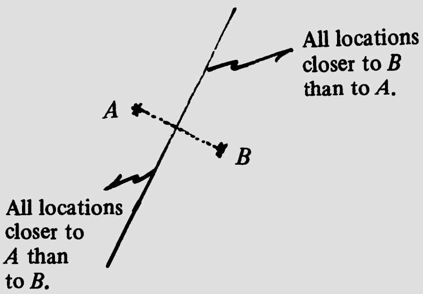


FIGURE 13-1 Dirichlet Domains of a pair of points

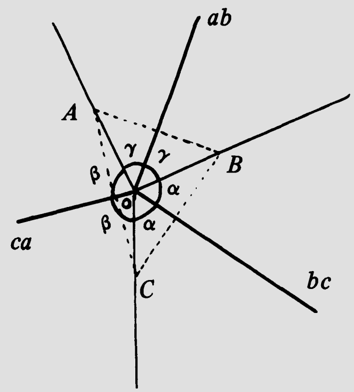


FIGURE 13-2 Dirichlet Domains of three points

My students raised the following question: Suppose that an ancient map of church parishes were found; would it be possible to discover whether these parishes were indeed laid out as Dirichlet Domains, and, if so, would it be possible to compute the location of the original parish churches? The answer to this question can be found in Fig. 13-2. If the point O is considered as the meeting point of three parishes, then the angles between ab, bc, and ca can be measured on the map; we shall call these angles a (between ca and ab), b (between ab and bc), and c (between bc and ca). The unknown angles are now α, β, and γ; these indicate the directions in which to travel into each parish toward the supposed location of the parish church. From Fig. 13-2,

$$\begin{aligned} \alpha + \beta \phantom{{}+\gamma} &= c \\ \beta + \gamma &= a \\ \alpha \phantom{{}+\beta} + \gamma &= b \end{aligned}$$

Since $a + b + c = 360°$:

$$\alpha = 180° - a; \quad \beta = 180° - b; \quad \gamma = 180° - c$$

By applying these expressions to each corner of every parish, lines can be drawn from these corners toward the potential location of the parish church. *If* these lines do indeed intersect in a single point for each parish, then every parish has the shape of a Dirichlet Domain, and one might initiate a "dig" at the points of congruence!

If it is desired to divide a plane into Dirichlet Domains belonging to *four* points, there is generally no single point, equidistant from all four points, where the four domains could all meet. Only in the exceptional case where the four given points happen to lie on a common circle would the four domains meet at a single point. In Fig. 13-3 six lines are drawn, perpendicularly bisecting each of the *six* connections between the *four* points *A, B, C,* and *D.* These six bisectors generally intersect each other in four different points; each of these four points is equidistant from a particular set of three out of the four original points *A, B, C,* and *D,* and is labeled accordingly with three lower-case characters. In Fig. 13-3 the lines *ab* and *bc* constitute part of the domain boundaries around point *B.* However, their intersection, the point *abc,* is closer to *D* than to *B;* the bisector *bd* passes *between B* and *abc.*

The point *abc,* as a matter of fact, lies inside the domain of *D!* In general, the Dirichlet Domain of any point is bounded by the *innermost* track that can be constituted out of bisector lines surrounding the point. Note that, analogously, the point *acd,* equidistant from *A, C,* and *D,* lies inside the domain of *B,* to which it is closest. However, the points *abd* and *bcd* do represent meeting

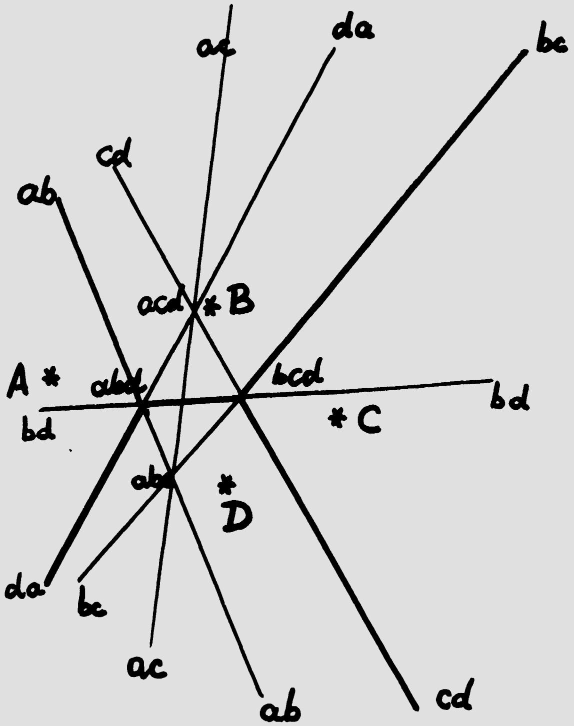


FIGURE 13-3 Dirichlet Domains of four points

points of the domains of the three points from which they are equidistant.

The concept of the Dirichlet Domain can be used to define *nearest neighbors* in a set of discrete points: *nearest*-neighbor points are those points *only* whose Dirichlet Domains share a boundary. In Fig. 13-3 the points *B* and *D* are nearest neighbors, whereas *A* and *C* are not. Of course, *A* and *B, B* and *C, C* and *D,* and *D* and *A* are also nearest-neighbor pairs.

Bici Pettit has used these principles to draw school districts in Cambridge, Massachusetts (Fig. 13-4). With a single exception three domains meet each other at each vertex. In one exceptional instance there is a quadruple meeting point. This quadruple point, being equidistant from *four* schools, lies at the center of a circle which happens to pass through four schools simultaneously.

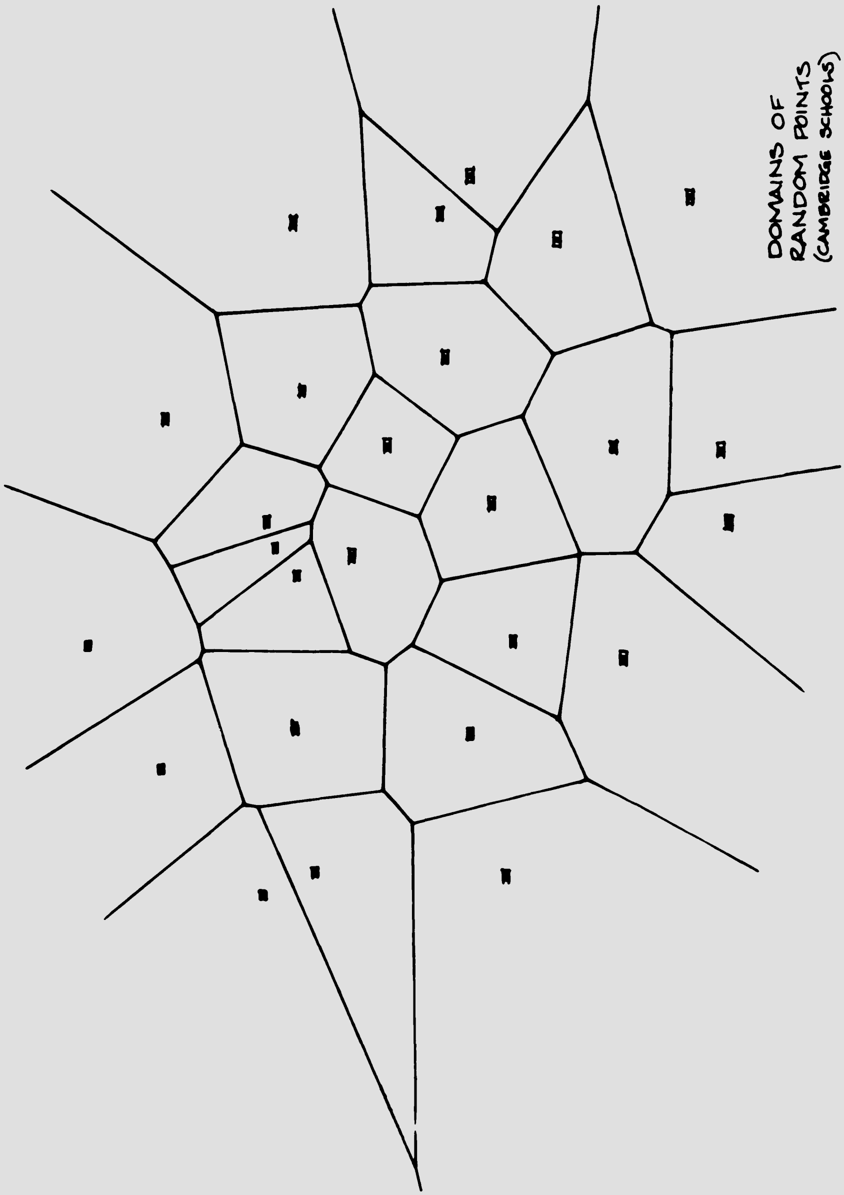
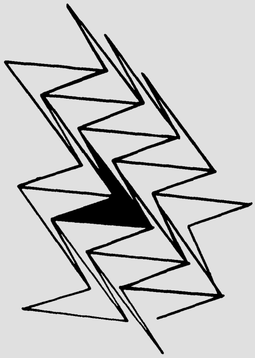


FIGURE 13-4 Dirichlet domains of Cambridge, Massachusetts, schools (drawn by Bici Pettit)



14

Dirichlet Domains of Regularly Spaced Planar Arrays

The infinitely extended plane may be subdivided into mutually *congruent* regions. We have already found restrictions on the valencies of the faces and vertices of such tessellations; if we wish all *vertices* to have identical valencies, and also require all *faces* to have identical valencies, we are limited to triangular, quadrilateral, and hexagonal faces, and respective vertex valencies of 6, 4, and 3. The further requirement that these faces be mutually congruent may impose additional restrictions, which we shall now investigate. It is easy to see that equilateral (equal edge length) faces having all angles equal to each other will satisfy the condition of filling the plane. However, this condition is more restrictive than necessary for congruency.

The plane can be filled with mutually congruent parallelograms, whose edge lengths need not all be the same, and whose angles are generally not all equal to each other. This does not imply, though, that parallelograms are Dirichlet Domains. By definition, Dirichlet Domains are plane fillers: any location within a Domain must lie closest to *some* Domain center than to all others, hence every point in the plane must belong to one Domain or the other. Although Dirichlet Domains are plane fillers, however, not all plane fillers are necessarily Dirichlet Domains. We shall therefore first examine some of the requirements that make polygons plane fillers, and next find the requirements that, in addition, make such polygons Dirichlet Domains.

The plane may be filled by mutually congruent parallelograms of any size or shape whatever. Every parallelogram may be divided diagonally into two triangles. Therefore, *every* triangle is a plane filler, for it can always be combined with its replica into a parallelogram which is in turn replicated to fill the plane. It is not so easy to see that *any* quadrilateral, not necessarily a parallelogram or square, is a plane filler. In Fig. 14-1 we show an arbitrary quadrilateral together with its replica rotated 180° about a point at the center of one of its edges. If points are chosen on the centers of edges of all quadrilaterals so generated, an infinite array of quadrilaterals results which covers the plane without leaving any gaps or overlaps (Fig. 14-2). Thus we have demonstrated that *any* quadrilateral is a plane filler. Figure 14-3 demonstrates, moreover, that the quadrilateral need not be convex.

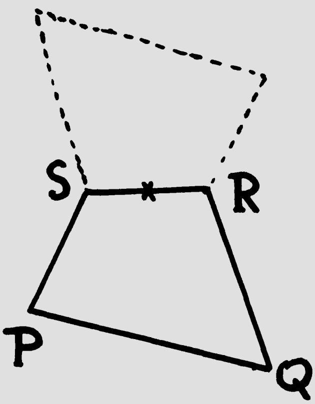


FIGURE 14-1
A quadrilateral

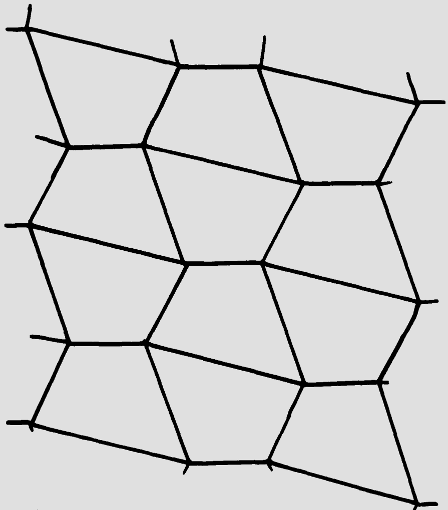


FIGURE 14-2
Quadrilateral plane filling

Since pairs of adjacent quadrilaterals in Fig. 14-2 together constitute a hexagon, arbitrary except for the requirement that its six edges constitute three parallel pairs of opposite edges of equal length, it is seen that such hexagons are also plane fillers.

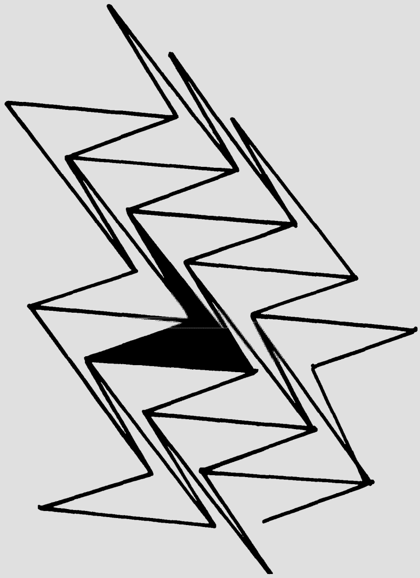


FIGURE 14-3 Quadrilateral plane filling: the quadrilateral does not need to be convex!

We shall now explore the additional constraints imposed by the requirement that the faces of the tessellations be Dirichlet Domains. From the construction of Dirichlet Domains demonstrated in Chapter 13 it follows that every Domain must contain a center such that every edge perpendicularly bisects a line joining centers in mutually adjacent Domains. For a general triangular tessellation the Domain centers may be chosen equidistant from the vertices of the corresponding Domain—i.e., at the intersection of the perpendicular bisectors of its edges (Fig. 14-4). In adjacent domains congruency of the triangles ensures equidistance of the centers from the common boundary; therefore the edges perpendicularly bisect a line joining adjacent Domain centers. Thus any triangle satisfies the requirement

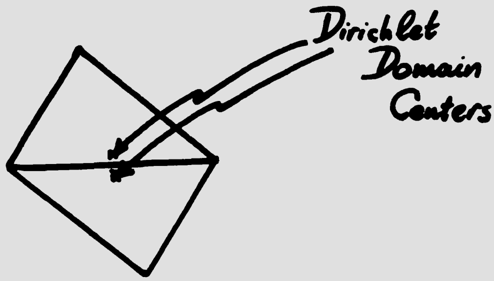


FIGURE 14-4 Adjacent triangular Dirichlet Domains

of a Dirichlet Domain, as long as the perpendicular bisectors of its edges intersect *inside* the triangle—in other words, as long as none of its angles exceeds 90°. We saw in Chapter 9 that a triangular tessellation requires 6-valent vertices, hence six triangular Dirichlet Domains must meet at each vertex.

The centers of an array of triangular Dirichlet Domains constitute vertices of a structure *dual* to that of the Domains (Fig. 14-5). Since the edges of the dual structures mutually perpendicularly bisect each other, the hexagonal faces defined by the centers of the triangular Domains also satisfy the requirements of Dirichlet Domains. Therefore these dual triangular and hexagonal structures bear the remarkable relationship to each other that the Domain centers of one constitute the vertices of the other, and vice versa. We observe that the triangular faces can be quite generally chosen, the only restriction being that the angles may not exceed 90°; the hexagonal faces necessarily have pairs of parallel edges of equal length, and their vertices must lie on a common circle whose center is the Domain center.

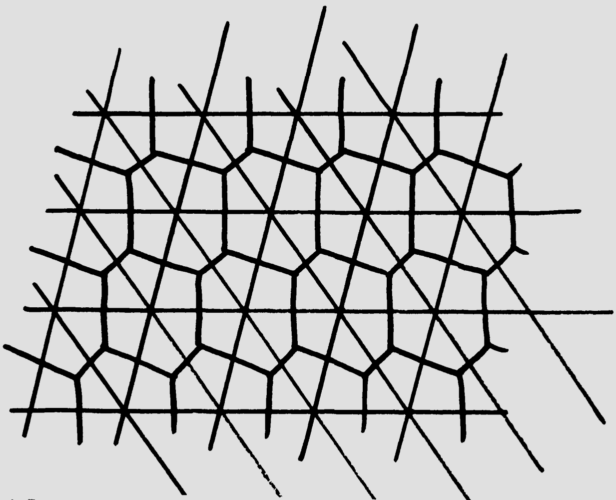


FIGURE 14-5
A triangular array of Dirichlet Domains, and its dual structure, a hexagonal array of Dirichlet Domains

Finally, we shall consider whether *any* quadrilateral can be used as a Dirichlet Domain. Figure 14-6 shows a pair of mutually congruent Dirichlet Domains, *PQRS* and *TUSR*, in plane-filling juxtaposition, their centers being A and B, respectively. We recall (cf. Fig. 14-1) that these two domains are related by a 180° rotation about a point at the center of line segment RS. The line AB therefore bisects RS. From the definition of Dirichlet Domains we already know that RS *perpendicularly* bisects AB. Since, therefore, AB bisects RS, and also is perpendicular to RS, AB and RS are shown to be each other's perpendicular bisectors. By repeating analogous arguments for each of the other domain boundaries, we prove that A must lie at the intersection of the perpendicular bisectors of its edges. The requirement that a quadrilateral be a Dirichlet Domain is therefore that its vertices lie on a common circle.

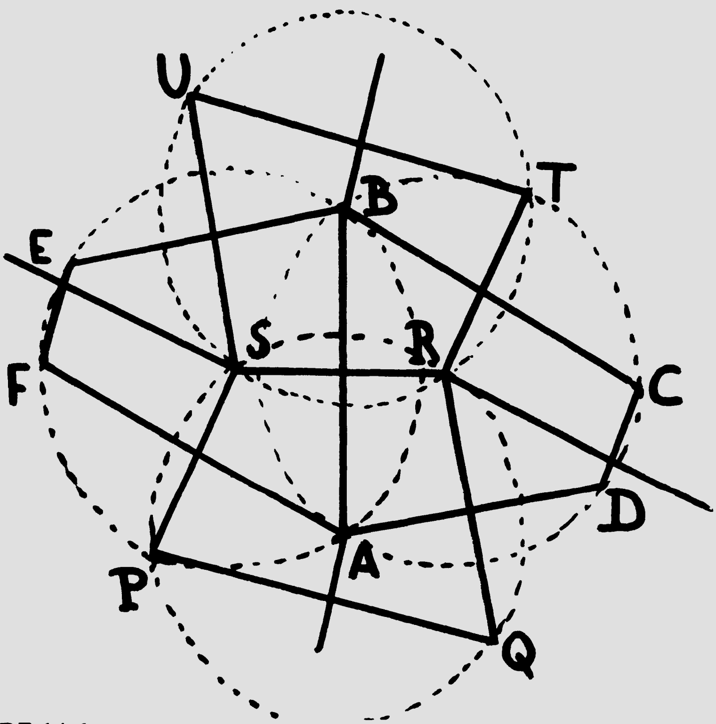
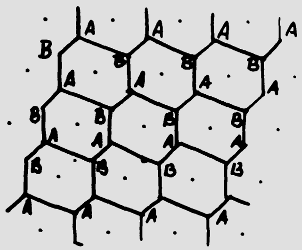


FIGURE 14-6
Quadrilateral Dirichlet Domains and their duals: *A* is the domain center for *PQRS*, while *R* is the domain center for *ABCD*

Again we note that the centers of a quadrilateral array of Dirichlet Domains (A, B, C, and D in Fig. 14-6) themselves constitute the vertices of a dual quadrilateral array, whose faces also satisfy the requirements for a Dirichlet Domain. We have thus shown that a polygon must satisfy the following requirements in order to be a Dirichlet Domain:

1. Its vertices must lie on a common circle.
2. Its edge valency is limited to values $n = 3, 4$, or 6.
3. If it is hexagonal, pairs of opposite edges must be parallel and equal in length.

Furthermore, we have shown that a tessellation of the plane into Dirichlet Domains has a dual tessellation whose vertices constitute the centers of the original Domains, and whose faces are Dirichlet Domains having as centers the vertices of the original Domains.



15

Lattices and Lattice Complexes

A *lattice* is an array of points each of which has identical environment in identical orientation. The points of a lattice are related to each other by a *translation:* by moving the entire lattice parallel to itself through an appropriate distance it can be brought into coincidence with itself (cf. Fig. 15-1). It follows that a lattice is infinite in extent. The points of any planar lattice may constitute the centers of hexagonal Dirichlet Domains; we saw in the previous chapter that such Domains necessarily have paired equal and parallel opposite edges (Fig. 15-2). Each lattice point therefore generally shares Dirichlet Domains with six other lattice points, hence has six near neighbors.

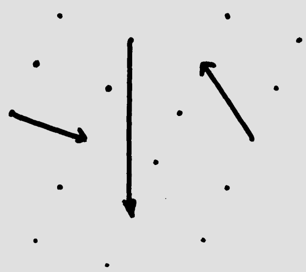


FIGURE 15-1
A lattice (the arrows denote displacements possible for the entire lattice to be brought into coincidence with itself)

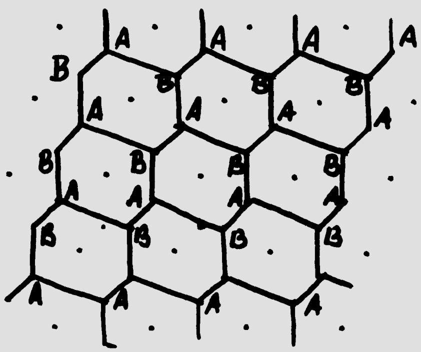


FIGURE 15-2 Lattice and its Dirichlet Domains (cf. Fig. 14-5)

The vertices of the hexagonal Dirichlet Domains do not together constitute a single lattice; they *do* have identical environments, but the orientation of the environment of half of the vertices is opposite to that of the other half. These vertices therefore belong to two distinct lattices, labeled A and B in Fig. 15-2. Points whose environments are identical except possibly for orientation are said to constitute a *lattice complex.* The points marked A and B in Fig. 15-2 together constitute a lattice complex.[1]

We showed in the previous chapter that Dirichlet Domains occur in dual pairs, the centers of one set constituting the vertices of the other set. This is also the case in Fig. 15-2: the points A and B constitute the centers of *triangular* Dirichlet Domains whose vertices constitute the original lattice. The triangles of this lattice were chosen arbitrarily. If they had been equilateral, their dual Dirichlet Domains would have been regular hexagons–i.e., hexagons whose edges have constant lengths and whose angles are all 120°.

If the lattice is specialized in that one of the angles of its triangles equals 90°, then the points A and B coincide on the triangles' hypotenuses (Fig. 15-3). The Dirichlet Domains of such a *rectangular* lattice are *rectangles* (the shortest sides AB of the hexagonal domain in Fig. 15-2 having vanished), so that each lattice point now has *four* instead of *six* nearest neighbors. The dual pairs of Dirichlet Domains are, for this special case, all mutually congruent rectangles.

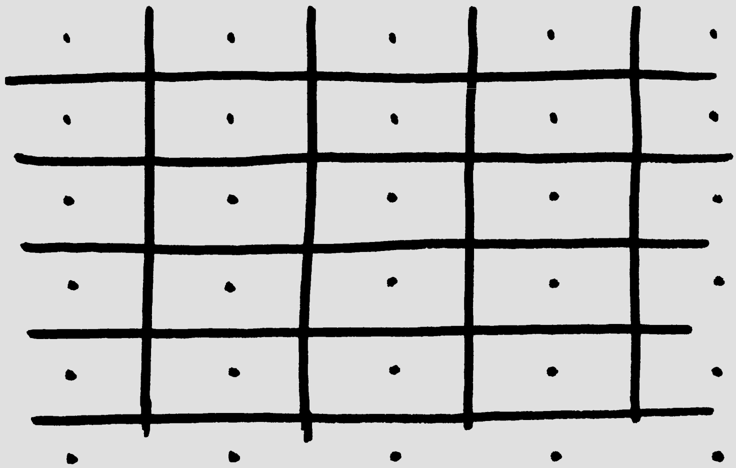
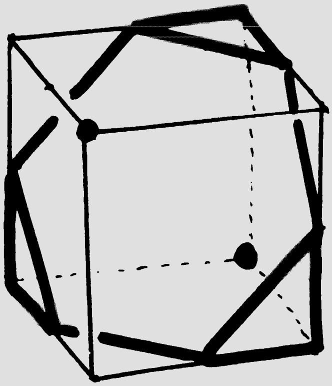


FIGURE 15-3 Rectangular lattice, rectangular Dirichlet Domains

[1] Cf. W. Fischer, H. Burzlaff, E. Hellner, and J. D. H. Donnay: *Space Groups and Lattice Complexes* (U.S. Department of Commerce, 1973).

If the triangle, in addition to having a right angle, is also isosceles, the Dirichlet Domains all become squares. We may distinguish between Dirichlet Domains whose shape is *regular* (equilateral triangle, square, or 120° equilateral hexagon), and those whose Dirichlet Domains are not. The difference between these classes is that the regular Domains have a uniquely fixed shape (only their *scale* is arbitrary), whereas the irregular domains have at least one variable–e.g., the ratio of two edge lengths or an angle, which can be set arbitrarily.

In three dimensions one may analogously define Dirichlet Domains appropriate to various lattices. Here the Domains are *cells* in space; by definition these cells must fill all of space, and, since their centers constitute a lattice, they must be mutually congruent. A cell whose replicas together may fill all of space is called a *space filler.* Space fillers are not necessarily *regular* polyhedra: the cube is the only regular space filler. The structure generated by filling space with cubes is also regular: all edges have $s = 4$, and all vertices have a valence of 6 toward *edges,* 12 toward faces, and 8 toward cells. In the next chapter we shall find other space fillers and the lattices with which they are associated.



16

Spacefilling Polyhedra

In the previous chapter we defined a space filler as a cell whose replicas together can fill all of space without having any voids between them. We saw that all Dirichlet Domains are space fillers, but that not all space fillers are necessarily Dirichlet Domains.

In the plane we found a triangle, a quadrilateral and a hexagon having paired parallel edges of equal length which may serve as Dirichlet Domains. We found three lattice complexes whose Domains have uniquely determined shapes without arbitrary parameters: a triangular lattice (hexagonal Domains), a square lattice (square Domains), and a hexagonal lattice *complex* (triangular Domains). Since several directions in such lattices and complexes are equivalent, they are called *isometric*.

There are three isometric three-dimensional lattices, which derive their names from the directions of the shortest vectors that bring them into coincidence with themselves (Fig. 16-1). In the *primitive cubic* lattice there are *six* such vectors, three mutually perpendicular pairs, directed along the edges of stacked space-filling cubes. In the *body-centered* lattice there are *eight* such vectors, directed toward the *centers* of eight cubes meeting at any lattice point. The body-centered lattice can be thought of as constituting both the vertices

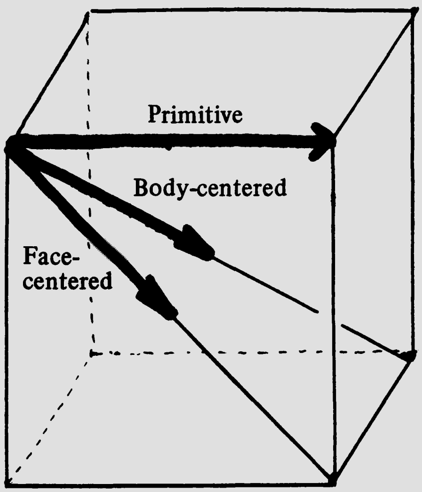


FIGURE 16-1 Translation vectors in primitive, body-centered, and face-centered lattices

and centers of stacked space-filling cubes; it is well to remember, however, that all these vertices are mutually equivalent, and that it is purely arbitrary which are considered vertices, and which cell centers.

In the *face-centered* lattice the shortest vectors that bring the lattice into coincidence with itself are directed from a vertex of stacked space-filling cubes to each of the centers of the *twelve* cube faces meeting at any vertex. Again, it should not be forgotten that all lattice points are equivalent, and that the reference to the cell and face centers of a set of stacked cubes is a purely artificial one, brought about by the crystallographers' preoccupation with the cube. We have seen that the cube, unlike the octahedron and tetrahedron, is not a stable configuration, hence not likely to be a natural building block. The face-centered lattice could equally well be described by the twelve vertices emanating from the center of a cube to the centers of each of its twelve edges; this description is entirely equivalent to the one given above, and would give rise to the name *edge-centered* rather than *face-centered* lattice. Much more fundamental is the relationship between each lattice and its associated Dirichlet Domain, which we shall now explore.

The Dirichlet Domain of a primitive cubic lattice is itself a cube. The *vertices* of the stacked cubes constitute a primitive cubic lattice dual to that constituted by the centers of the cubic cells: we have seen that a structure consisting of stacked cubic cells filling all of space is self-dual (Chapter 9).

When a lattice-point in the body-centered cubic lattice is joined to eight neighbors in the directions of the four body diagonals of the cube, and the joining lines are perpendicularly bisected, the bisecting planes together constitute an octahedron centered on that lattice point. However, the vertices of such an octahedron lie closer to six other lattice points than to the central one; therefore the octahedron fails as a Dirichlet Domain (Fig. 16-2). Recalling that the Dirichlet Domain is the *innermost* region enclosed by perpendicular bisectors of edges joining *all* neighbors, we bisect perpendicularly also the six edges in the primitive cubic lattice (along the direction of cube edges). These bisector planes truncate the octahedron previously obtained: the Dirichlet Domain of the body-centered cubic lattice is a truncated octahedron whose edges all have equal edge length, whose six quadrilateral faces are squares, and whose eight hexagonal faces are regular. Such a truncated octahedron, being a Dirichlet Domain, is accordingly a space filler. It is a semiregular polyhedron: all vertices are equivalent. Notable also is that in the space-filling

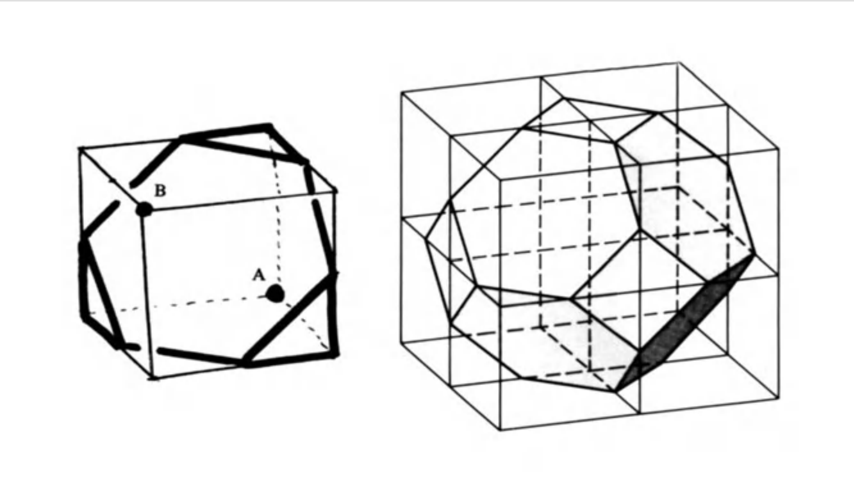


FIGURE 16-2a The hexagon lies in a plane perpendicularly bisecting a line joining points *A* and *B* of a body-centered cubic lattice. Thus one octant of the Dirichlet Domain is formed.

FIGURE 16-2b When a point in a body-centered cubic lattice is joined to eight other points located at cubic vertices around the central point, eight octants like Fig. 16-2a are joined to form a cuboctahedron.

FIGURE 16-2 Construction of a Dirichlet Domain of the body-centered cubic lattice.

structure made up of truncated octahedron all *edges* are mutually equivalent. Their valency s equals 3. Whereas in an individual truncated octahedron there are two distinct kinds of edges—namely, those joining a square to a hexagon and those joining two hexagons together—the three-dimensional structure is constituted such that *each* edge joins *two* hexagonal faces and *one* square one. Thus, in the truncated-octahedron space filling all cells are mutually equivalent, as are all vertices and all edges, but there are *two* kinds of faces.

From the equations derived in Chapter 4 we can find the valencies describing truncated-octahedron space filling. For the truncated octahedron $k = 14, l = 36, m = 24$. Hence, from equation (4-13): $\bar{n} = 5^1/_7$. From equation (4-14): $p/q = 3/2$. From equation (4-11): $p = 6$, hence $q = 4$; and, from equation (3-5): $r = 4$. Thus we see that all vertices have the lowest valencies possible in a three-dimensional structure; this type of space filling and the body-centered cubic lattice constituted by the centers of the cell therefore are very common.[1] The vertices of the space-filling structure are mutually equivalent, but their environments are in different orientations. They do not constitute a lattice, but a *lattice complex* whose importance has been stressed by Hellner *et al.*[2] The body-centered cubic lattice, when each lattice point is joined to its fourteen near neighbors by edges perpendicular to its fourteen Domain faces, is dual to the above-mentioned lattice complex in the sense of three-dimensional duality discussed in Chapter 7.

When a lattice point in the *face*-centered lattice is joined to its twelve nearest neighbors in the face centers, and the joining edges are bisected, a Dirichlet Domain results in the form of a rhombohedral dodecahedron, having $k = 12, m = 14, l = 24, n = 4$. This dodecahedron is, accordingly, a space filler, whose centers constitute the face-centered cubic lattice. All faces are mutually equivalent, as are all edges, which have $s = 3$. From equation (4-11), $\bar{p} = 8$; from equation (4-12), $\bar{r} = 5^1/_3$; and from equation (4-14): $\bar{q} = 4^2/_3$. The dodecahedral space-filling structure has two types of vertices, corresponding to the 4-valent and 3-valent vertices of the dodecahedral cell. We recall that the rhombohedral dodecahedron is the degenerately stellated form of the cube and the octahedron. Accordingly, we can visualize the dodecahedral space-filling structure in relation to

[1] Cf. Cyril S. Smith: in *Hierarchical Structures,* L. L. Whyte, A. G. Wilson, and D. Wilson, eds. (American Elsevier, New York, 1969).

[2] Cf. W. Fischer, H. Burzlaff, E. Hellner, and J. D. H. Donnay: *Space Groups and Lattice Complexes* (U.S. Department of Commerce, 1973).

the cubic one as follows. Every cubic cell is surrounded by six adjoining adjacent cells. Each of these six cells may be subdivided into six pyramids, each having as base a cube face, as apex the center of the cubic cell.[3] By subdividing half of the cubic cells in a cubic space-filling structure into six pyramids, and adding these pyramids to the adjacent intact cubic cells, we stellate the latter into dodecahedral cells, creating the dodecahedral space-filling structure (Fig. 16-3). The volume of each cell exactly equals twice that in the cubic space-filling structure. The dodecahedral structure has vertices at the

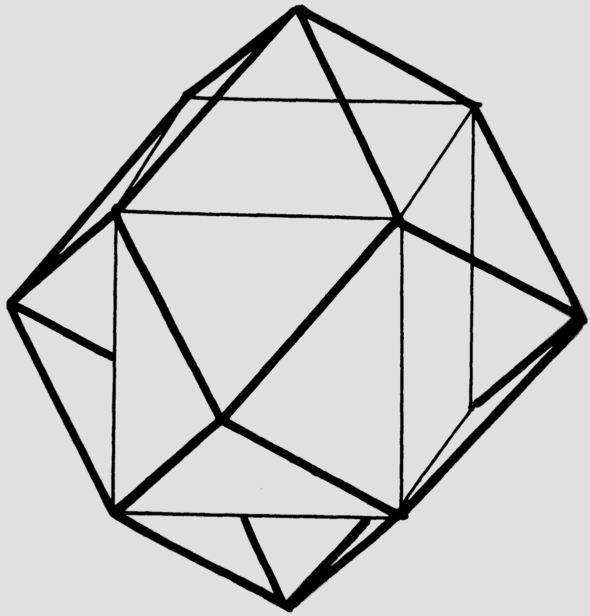


FIGURE 16-3 Cubic stellation to dodecahedron

centers of the former cubic cells; for these vertices, accordingly, $r = 8$, $q = 6$, $p = 12$. Remembering that the rhombohedral dodecahedron is also a stellated form of the octahedron, we can consider the rhombohedral dodecahedral cell as made up of an octahedron with eight triangular pyramids added to its faces to accomplish the stellation. Four of such pyramids, when joined at their apex, constitute a regular tetrahedron. Accordingly, those vertices corresponding to the 3-valent vertices of the rhombohedral dodecahedral cell could also be considered as the centers of tetrahedral cells in combined octahedron/tetrahedron packing. Accordingly, these vertices have,

[3] Cf. "Coupler," in R. Buckminster Fuller, with E. J. Applewhite and A. L. Loeb: *Synergetics* (Macmillan, New York, 1975), pp. 541-549.

in dodecahedral space filling, $r = 4$, $q = 4$, $p = 6$. In a dodecahedral space-filling structure we have accordingly cubically coordinated vertices having $r = 8$, $q = 6$, $p = 12$, and tetrahedrally coordinated vertices having $r = 4$, $q = 4$, $p = 6$. To arrive at the average values computed above to be $r = 5^1/_3$, $q = 4^2/_3$, $p = 8$, we conclude that the tetrahedrally coordinated vertices must be twice as numerous as the cubically coordinated ones.

We have thus established a connection between the lattices so familiar to the crystallographer and the space-filling polyhedra. The significance of the vertices of these polyhedra in systematic crystallography has been pointed out by the author.[4]

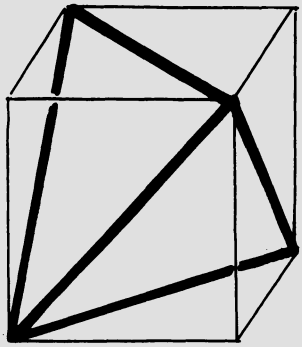
Other polyhedra significant in space structures are the *coordination polyhedra:* these describe the configuration made by nearest neighbors around a central lattice point. A coordination polyhedron is defined as a polyhedron whose center is a lattice (complex) point, and whose vertices are the centers of Dirichlet Domains sharing a face with the Domain of the given lattice (complex) point. In a primitive cubic lattice the Dirichlet Domain is a cube, which shares *six* faces with adjacent Domains. Accordingly, the coordination polyhedron in a primitive cubic lattice has six vertices: it is an octahedron.

In a body-centered cubic lattice the Dirichlet Domain is a truncated octahedron: there are fourteen nearest neighbors, of which six share a square face, eight a hexagonal face with the central cell. The coordination polyhedron is, accordingly, a rhombohedral dodecahedron, which thus functions as coordination polyhedron in the body-centered cubic lattice as well as a Dirichlet Domain in the face-centered cubic lattice.

The face-centered cubic lattice has a twelve-faced Dirichlet Domain. Its coordination polyhedron is the cuboctahedron, which has twelve vertices. It should be noted that in the primitive and in the face-centered cubic lattice the Dirichlet Domain and coordination polyhedron are each other's duals, but in the body-centered lattice they are *not.* It is therefore *not* generally true that Dirichlet Domain and coordination polyhedron are dually related.

The possibility of filling spaces with a combination of octahedral and tetrahedral cells was alluded to above. For a further discussion of such mixed space filling, cf. A. L. Loeb, "Contributions to R. Buckminster Fuller's *Synergetics,*" pp. 836-855 (Macmillan, 1975).

[4] A. L. Loeb: *J. Solid State Chem. 1,* 237-267 (1970).



17

Additional Space Fillers and their Lattice Complexes

The primitive, body-centered, and face-centered lattices are the only isometric ones in three dimensions: their Dirichlet Domains–respectively, the cube, truncated octahedron, and rhombohedral dodecahedron–fill space; all three have the maximum symmetry. There are, in addition, interesting lattice *complexes* whose Dirichlet Domains also, of course, fill space. Since the environments of lattice-complex points are identical, but not necessarily oriented parallel, their Dirichlet Domains will combine in various orientations to fill space, and have lower symmetry than those of lattice points.

First consider the so-called J-complex. Again it derives its name from the crystallographers' preoccupation with the cube as a frame of reference: the points of a J-complex occupy the *centers of faces* (but *not* the vertices) of space-filling stacked cubes, thus assuming the familiar jackstone configuration (Fig. 17-1).

In this case the reference to stacked cubes provides a real convenience. If *each* of the cubes is divided into six pyramids meeting at the cube center, each having as base a square face of the cube, then each of the points of the J-complex lies in the base of one of these pyramids. Each point of the J-complex thus finds itself at the center of a *square bipyramid,* which happens to be its Dirichlet Domain.

This Domain is an *octahedron,* but not a *regular* one; it is, however, a space filler. The geometry of this space-filling octahedron is quite interesting, and merits some special attention. Whereas the *regular* octahedron has three equally long body diagonals, the space-filling octahedron has one body diagonal shorter than the other two. Recalling that the space-filling octahedron consists of two square pyramids, each constituting one-sixth of a cube, we see that the length of its shortest body diagonal equals the edge length of the cube from which it was generated, which also equals the length of those edges of the space-filling octahedron that are perpendicular to this shortest diagonal. If we call this length *unity,* and realize that the other two

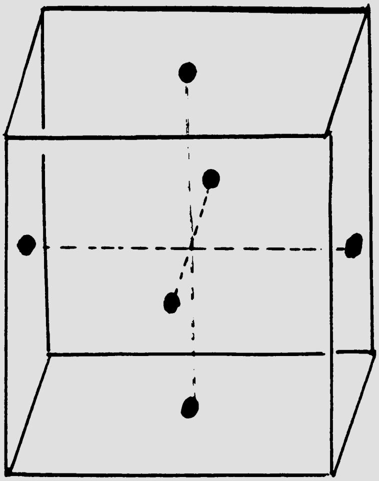


FIGURE 17-1 **Points of the J-lattice complex related to the cube**

body diagonals of this space-filling octahedron are the *face* diagonals of the generating cube, we find for the ratio of the lengths of the body diagonals of the space-filling octahedron the value $1 \div \sqrt{2}$. The four edges perpendicular to the shortest diagonal have length unity; the other eight, being half of the body diagonals of the generating cube, have length $^1/_2 \sqrt{3}$. To fill space with these octahedra they have to be stacked in three different orientations, with their shortest body diagonals mutually perpendicular. Recalling the construction of the rhombohedral dodecahedron by degenerate stellation of the cube, we realize that six space-filling octahedra, meeting at one end of this short diagonal, together form a rhombohedral dodecahedron, each rhombic face containing two triangular faces of the octahedron.[1]

The angles of these space-filling polyhedra are also of interest. The rhombohedral dodecahedron has acute angles at its 4-valent vertices, and obtuse ones at its 3-valent vertices. Since the three edges meeting at a 3-valent vertex are not coplanar, but make equal angles with each other, each obtuse angle is less than 120°, but greater than 90°. The acute angle is the smaller angle between two body diagonals of the cube. Since the acute and obtuse angles in each rhombic face of the dodecahedron must add up to 180°, these angles in fact are the smaller and larger angles between body diagonals of the cube. In Fig. 17-2 we show a diagonal cross section of the cube, including the intersection of its body diagonals. If we call the smaller angle between these δ, it follows that

$$\sin\left(\frac{1}{2}\,\delta\right) = \left(\frac{1}{2}\right) / \left(\frac{1}{2}\sqrt{3}\right)$$

$$\therefore\ \cos\delta = 1 - 2\sin^2\left(\frac{1}{2}\,\delta\right) = \frac{1}{3}$$

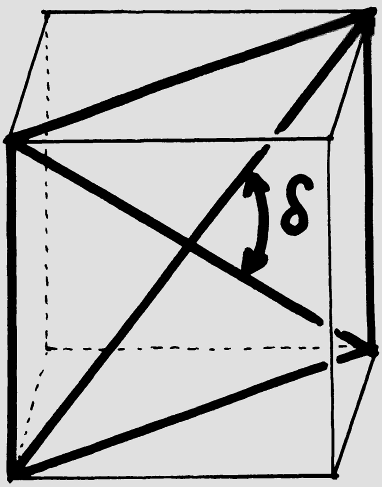


FIGURE 17-2 Diagonal cross-section of the cube showing the intersection of two body diagonals

[1] Cf. "Coupler," in R. Buckminster Fuller, with E. J. Applewhite and A. L. Loeb: *Synergetics* (Macmillan, New York, 1975), pp. 541ff.

The angle δ equals, in fact, a bit less than 71°, but it is more important to recall it as the angle whose cosine equals *one-third.* The angles of the rhombohedral dodecahedron are therefore those whose cosines equal $+ \frac{1}{3}$ and $- \frac{1}{3}$; the angles of the space-filling octahedron at each end of the short diagonal also have their cosine equal to $\frac{1}{3}$. (Of course, the angles of the *regular* octahedron have their cosine equal to $\frac{1}{2}$.)

It is recalled that degenerate stellation of the tetrahedron yields a hexahedron–i.e., in the *regular* case a cube. This relation of the tetrahedron to the cube is shown in Fig. 17-3; the center of this tetrahedron coincides with that of the cube. It has been shown that the tetrahedron occupies exactly one-third of the volume of the cube in which it is inscribed.[2] From this juxtaposition of cube and tetrahedron one sees that the directions of lines joining the center of a tetrahedron to its vertices are the same as those of the body diagonal of the cube. The angle subtended by its vertices at the center of a regular tetrahedron is therefore also the one whose cosine equals $- \frac{1}{3}$. When lines joining the vertices of a regular tetrahedron to its center are extended until they meet the faces of the tetrahedron, they meet these faces perpendicularly. The angle between the faces themselves is therefore the angle δ, whose cosine equals $+ \frac{1}{3}$. This angle, besides 90° and 60°, occurs and re-occurs in space structures. Because of the unfortunate choice of the unit of angle, it does not have a nice round value, with the result that it is not as familiar as

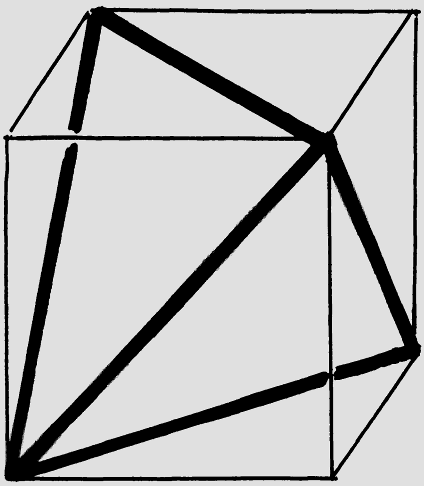


FIGURE 17-3 Tetrahedron inscribed in a cube

the other two. This lack of familiarity has been a handicap to the designer; it is hoped that this discussion illuminates some properties of this important angle.

The proof that the inscribed tetrahedron occupies one-third of the volume of the cube depended on the realization that the four portions of the cube *outside* this tetrahedron are octants of the octahedron.[2] Regular octahedra and tetrahedra of equal edge length together fill space in numerical ratio $1 \div 2$; since a cube is a space filler and consists of a tetrahedron plus a half-octahedron, this ratio is evident. Analogously, the rhombohedral dodecahedron is a degenerate stellated octahedron, consisting of an octahedron with eight triangular pyramids stuck on its faces; four of such pyramids together constitute a regular tetrahedron. Thus again the ratio of one octahedron to two tetrahedra prevails, because the rhombohedral dodecahedron fills space, and consists of *one* octahedron plus eight quarter-tetrahedra.

Since regular octahedra and tetrahedra combine to fill space, and since a space-filling, *non*-regular octahedron has also been found, one naturally wonders if a non-regular space-filling tetrahedron also exists. That the answer is affirmative can be seen by dividing the space-filling octahedron into four mutually congruent tetrahedra which join along the short diagonal of the octahedron. The short diagonal of the space-filling *octahedron* and an edge perpendicular to it constitute two edges of equal length of the space-filling tetrahedron. The remaining four edges of the tetrahedron are also of equal length, but $\frac{1}{2}\sqrt{3}$ times as long as (in other words, shorter than) the first two. The angles are, again, δ and $(90° - \frac{1}{2}\delta)$.

The irregular octahedron does *not*, by filling space, form a regular structure, for there are two types of vertices. Those at the ends of the short body-diagonal will, in the space-filling structure, have valencies $r = 8$, $p = 12$, $q = 6$. Those at the corners of the square equators have: $r = 14$, $p = 24$, $q = 12$. Since there are equal numbers of each type of vertex (the crystallographer would call this the caesium chloride structure, in which each vertex is surrounded by eight vertices of the other type, at the corners of a cube), $\bar{r} = 11$, $\bar{p} = 18$, $\bar{q} = 9$. The cell valencies are $n = 3$, $m = 6$, $l = 12$, $k = 8$, and hence $\bar{s} = 36/11$. In the space filling structure the edges converging at the

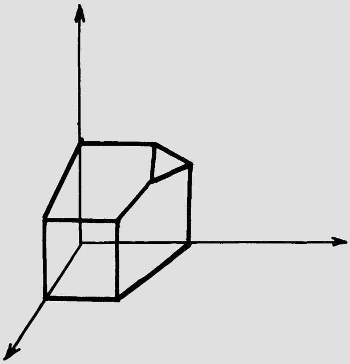
[2] A. L. Loeb, "Contributions to R. B. Fuller's *Synergetics*" (Macmillan, New York, 1975), pp. 832-836.

vertices having $r = 8$ have $s = 3$; the others have $s = 4$. Therefore the two types of edges occur in the ratio of 8 of the former to 3 of the latter.

Finally, a lattice complex of special interest is the D-complex, which derives its name from the *d*iamond structure. This structure is characterized by the fact that each of its points is surrounded by four nearest neighbors in directions making equal angles with each other. To visualize its Dirichlet Domain, consider a truncated tetrahedron: it has four hexagonal and four triangular faces (cf. Chapter 10), and in this case all its edges are equal in length. This polyhedron is *not* a space filler, hence by itself does not constitute a Dirichlet Domain. It combines with tetrahedra of equal edge length to fill space, the tetrahedra sharing their faces with the triangular faces of the truncated tetrahedron.[3] Another way of looking at this space-filling mode is to have truncated tetrahedra stacked with their hexagonal faces in contact; their centers constitute the D-lattice complex, but there are tetrahedron-shaped interstices between them. We have seen that a regular tetrahedron may be subdivided into four triangular pyramids whose apices are at the center of the tetrahedron. If we allocate a quarter of each tetrahedral interstice to each of the four truncated tetrahedra between which the interstice is located, we in fact stellate the *triangular* faces of the truncated tetrahedra to make them into Dirichlet Domains.[4]

[3] A. L. Loeb, "Contributions to R. B. Fuller's *Synergetics*" (Macmillan, New York, 1975), pp. 842, 843.

[4] Cf. A. L. Loeb, *J. Solid State Chem. 1*, 237-267 (1970); R. Buckminster Fuller, with E. J. Applewhite and A. L. Loeb: *op. cit.*, p. 874.



18

Orthorhombic and Tetragonal Lattices

In the previous chapters we discussed isometric lattices and their space-filling Dirichlet Domains. When these lattices are distorted by stretching the translation vectors differently in different directions, the Domains could be proportionately distorted, and still be space fillers. The Domains of the lattices would still be mutually congruent. However, the Domains of the lattice *complexes* would be stretched or compressed differently if originally they had been differently oriented. For instance, the space-filling octahedron has to be oriented in three different directions to fill space. If the spacing of the points of the J-complex, being at the centers of the octahedra, is altered differently in different directions, then some octahedra will be squashed or stretched along their polar axis, while others have their equators deformed instead.

More important, however, these space-filling Domains will no longer be Dirichlet Domains of the corresponding lattices, because their faces are no longer perpendicular to the edges joining lattice points.

We shall here examine what happens to the Dirichlet Domains of the isometric lattices when they are distorted differently in three mutually perpendicular directions, thus becoming orthorhombic. These lattices have two arbitrary parameters: the ratios of lattice spacings in the mutually perpendicular directions. The principal reason for our present interest in these particular lattices (the orthorhombic system does have additional lattices) is the transformation which the Dirichlet Domains experience as the arbitrary parameters are varied.

First, let us consider a primitive orthorhombic lattice in a Cartesian coordinate system so that there is a lattice point at the origin $(0, 0, 0)$, and the nearest six lattice points are at $(\pm 2a, 0, 0)$, $(0, \pm 2b, 0)$ and $(0, 0, \pm 2c)$, where $a \leqslant b < c$. Twelve additional nearby lattice points occur at $(\pm 2a, \pm 2b, 0)$, $(\pm 2a, 0, \pm 2c)$, and $(0, \pm 2b, \pm 2c)$; these may or may not affect the shape of the Dirichlet Domain. Planes equidistant from the origin and from its nearest neighbors have the equations $x = \pm a$, $y = \pm b$, and $z = \pm c$. They constitute a rectangular prism. The additional twelve points will contribute to the Dirichlet Domain if the bisector planes of straight lines joining each to the origin penetrate inside said prism. The coordinates of any point (x, y, z) equidistant from the origin and from $(2a, 2b, 0)$ obey the equation

$$x^2 + y^2 + z^2 = (x - 2a)^2 + (y - 2b)^2 + z^2$$

That is,

$$ax + by = a^2 + b^2$$

This plane just touches, but does not penetrate, the rectangular prism, so that lattice point $(2a, 2b, 0)$ does not directly affect the Dirichlet Domain. By a similar argument all other points besides $(\pm 2a, 0, 0)$, $(0, \pm 2b, 0)$ and $(0, 0, \pm 2c)$ are eliminated: the inner polyhedron formed by the perpendicular bisector planes is the abovementioned rectangular prism, which transforms to a square prism when $a = b$, and to a cube when $a = b = c$.

The body-centered orthorhombic lattice has fourteen near lattice points at $(0, 0, 0)$, $(\pm 2a, 0, 0)$, $(0, \pm 2b, 0)$, $(0, 0, \pm 2c)$, and $(\pm a, \pm b, \pm c)$. It is recalled from the isometric system that, certainly when $a = b = c$, the planes perpendicularly bisecting the edges joining $(0, 0, 0)$ to $(\pm a, \pm a, \pm a)$ do penetrate and are penetrated by the bisector planes $x = \pm a$, $y = \pm a$, and $z = \pm a$. Accordingly, we would expect that all fourteen lattice points listed above would figure in the constitution of the Dirichlet Domain.

Because of the equivalence of all eight octants meeting at the origin, we need only be concerned with a single octant, from which the entire Domain can be generated by mirror reflection in the coordinate planes. We shall therefore consider the perpendicular bisectors of the edges joining (0, 0, 0) to (a, b, c), to $(2a, 0, 0)$, to $(0, 2b, 0)$, and to $(0, 0, 2c)$. The equation of the first of these planes is

$$ax + by + cz = \frac{1}{2}(a^2 + b^2 + c^2)$$

Table 18-1 lists the intersections of this plane with the coordinate axes and with the edges of the prism whose faces are $x = a$, $y = b$, $z = c$, and the coordinate planes. All these twelve points of intersection are potential vertices of a Dirichlet Domain, but their candidacy for that function must be eliminated if *any one* of the following conditions exists for its coordinates:

$$x < 0, \quad y < 0, \quad z < 0, \quad x > a, \quad y > b, \quad z > c$$

Of the twelve points listed in Table 18-1, four are definitely eliminated. Two are always possible. The remaining six depend on the critical value of c^2 with respect to $a^2 + b^2$. If $c^2 < a^2 + b^2$, two of the six are eliminated, leaving a total of six vertices (Fig. 18-1); if

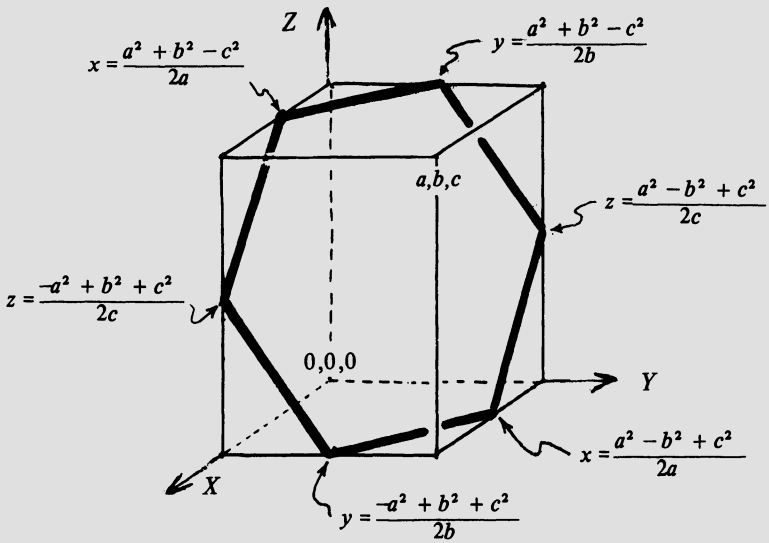


FIGURE 18-1 Dirichlet Domain vertices when $c^2 < a^2 + b^2$

TABLE 18-1
Potential Dirichlet Domain Vertices

	Intersection with bisector plane			
Line	*X-coordinate*	*Y-coordinate*	*Z-coordinate*	*Comment*
$x = y = 0$	0	0	$\frac{1}{2}(a^2 + b^2 + c^2)/c$	$z > c$ if $c^2 < a^2 + b^2$
$x = z = 0$	0	$\frac{1}{2}(a^2 + b^2 + c^2)/b$	0	$y > b$: not possible
$y = z = 0$	$\frac{1}{2}(a^2 + b^2 + c^2)/a$	0	0	$x > a$: not possible
$x = 0, y = b$	0	b	$\frac{1}{2}(a^2 - b^2 + c^2)/c$	Always possible
$x = 0, z = c$	0	$\frac{1}{2}(a^2 + b^2 - c^2)/b$	c	$y < 0$ if $c^2 > a^2 + b^2$
$y = 0, z = c$	$\frac{1}{2}(a^2 + b^2 - c^2)/a$	0	c	$x < 0$ if $c^2 > a^2 + b^2$
$x = a, y = 0$	a	0	$\frac{1}{2}(-a^2 + b^2 + c^2)/c$	Always possible
$x = a, z = 0$	a	$\frac{1}{2}(-a^2 + b^2 + c^2)/b$	0	$y > b$ if $c^2 > a^2 + b^2$
$y = b, z = 0$	$\frac{1}{2}(a^2 - b^2 + c^2)/a$	b	0	$x > a$ if $c^2 > a^2 + b^2$
$x = a, y = b$	a	b	$\frac{1}{2}(-a^2 - b^2 + c^2)/c$	$z < 0$ if $c^2 < a^2 + b^2$
$x = a, z = c$	a	$\frac{1}{2}(-a^2 + b^2 - c^2)/b$	c	$y < 0$: not possible
$y = b, z = c$	$\frac{1}{2}(a^2 - b^2 - c^2)/a$	b	c	$x < 0$: not possible

$c^2 > a^2 + b^2$, four of the six are eliminated, leaving a total of four vertices (Fig. 18-2). In the critical case $c^2 = a^2 + b^2$ the six of the latter eventuality merge into the four of the former (Fig. 18-3). We

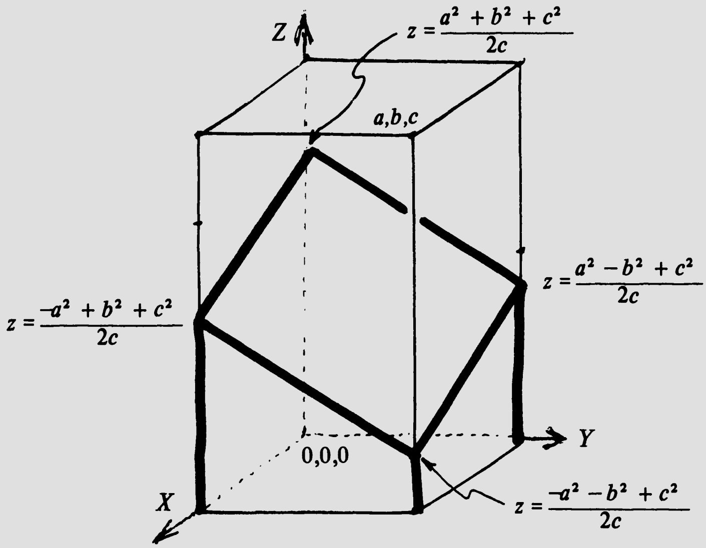


FIGURE 18-2 Dirichlet Domain vertices when $c^2 > a^2 + b^2$

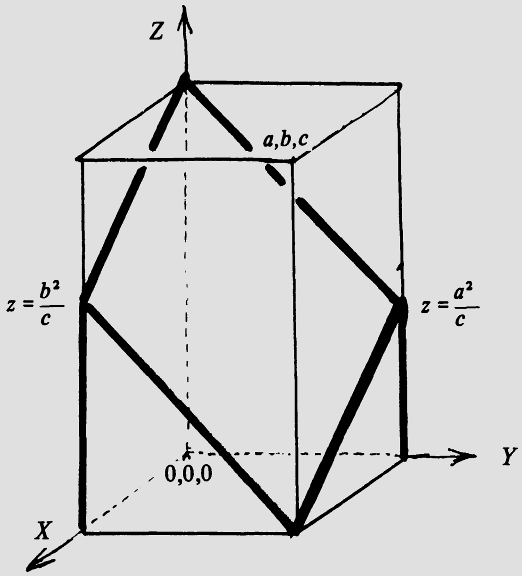


FIGURE 18-3 Dirichlet Domain vertices when $c^2 = a^2 + b^2$

conclude that for the range $c^2 < a^2 + b^2$, which includes the cube as a special case, the Dirichlet Domain of the body-centered orthorhombic domain is a truncated octahedron. For the range $c^2 > a^2 + b^2$ the Dirichlet Domain has eight rhombic faces (one in each octant), and four hexagonal ones, all four parallel to the Z-axis. At the critical value $c^2 = a^2 + b^2$ the number of nearest neighbors thus changes from fourteen to twelve; the two that vanish are the ones along the Z-axis, which become too distant when c increases.

In the critical case when $c^2 = a^2 + b^2$, the Dirichlet Domain has twelve *rhombic* faces: it is a rhombohedral dodecahedron! If $a^2 = b^2$, we have a special orthorhombic lattice, called *tetragonal.* This lattice, with $c^2 = 2a^2$, has as Dirichlet Domain just exactly the Dirichlet Domain found for the face-centered cubic lattice.

To reconcile the fact that a tetragonal lattice having $c^2 = 2a^2$ is synonymous with a face-centered cubic lattice, it is advisable to rotate one through 45° with respect to the other around the Z-axis. The body-centered lattice points in the tetragonal lattice then become face-centered in the cubic one, and the base of the square prism in the tetragonal lattice becomes transformed into the base of the face-centered cube (Fig. 18-4).

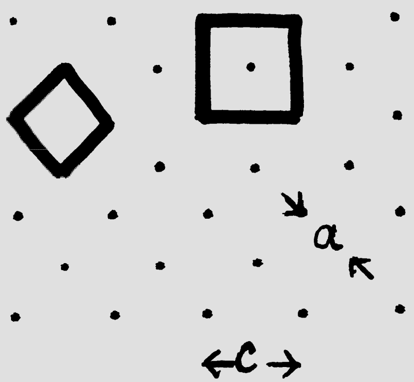


FIGURE 18-4 Bases of body-centered tetragonal and face-centered cubic unit cells

When $c^2 > a^2 + b^2$ (Fig. 18-2), the Dirichlet Domain has twelve faces, but four of these are hexagonal. Such a Dirichlet Domain could be considered as a rhombohedral dodecahedron bisected in an equator, with a rectangular prism inserted at the bisector. Since the dodecahedron and the prism separately are space fillers, there is no problem about filling space with the combination.

The Dirichlet Domain for the *face*-centered orthorhombic lattice, whose lattice points are at (0, 0, 0), (2a, 0, 0), (a, b, 0), (a, 0, c), (0, b, c), etc., is shown in Fig. 18-5. It is a truncated octahedron in a peculiar orientation: two quadrilateral faces are perpendicular to the X-axis, having edges parallel to the Y- or Z-axes. The other four quadrilateral faces are diagonally and perpendicularly bisected by the YZ-plane.

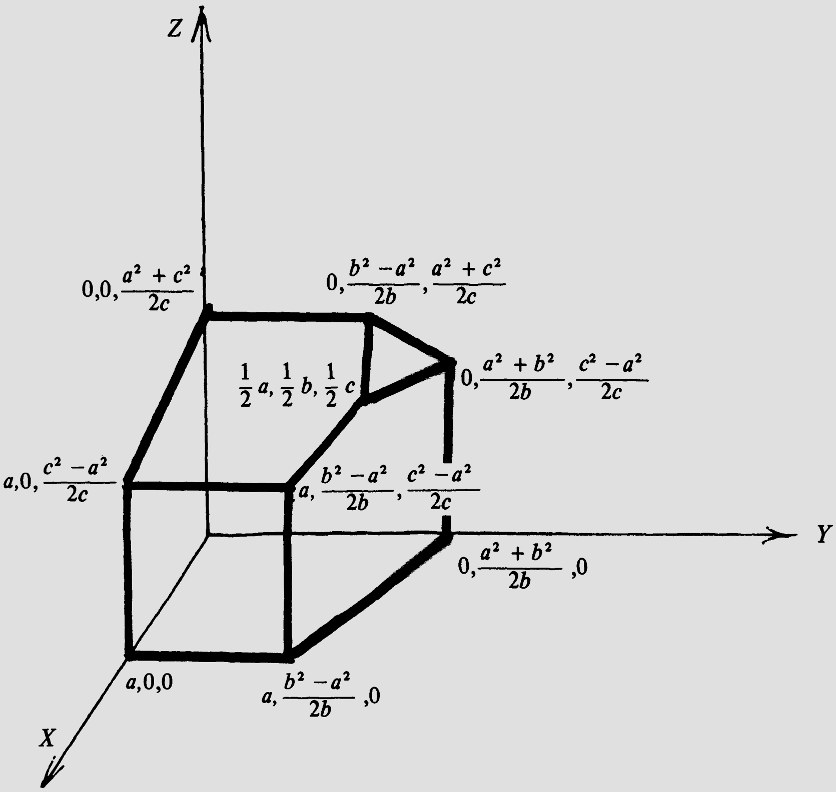


FIGURE 18-5 Dirichlet Domain of face-centered orthorhombic lattice

The vertices of the Domain are each equidistant from four of the lattice points. The vertex ($\frac{1}{2}a$, $\frac{1}{2}b$, $\frac{1}{2}c$) is equidistant from the three other vertices to which it is joined. Each face-centered orthorhombic lattice point has generally fourteen near neighbors.

For the special tetragonal case ($a = b$), the Dirichlet Domain becomes again the rhombododecahedron/prism combination having four hexagonal and eight quadrilateral (rhombic) faces (Fig. 18-6), and for $a = b = c$ becomes the dodecahedron itself.

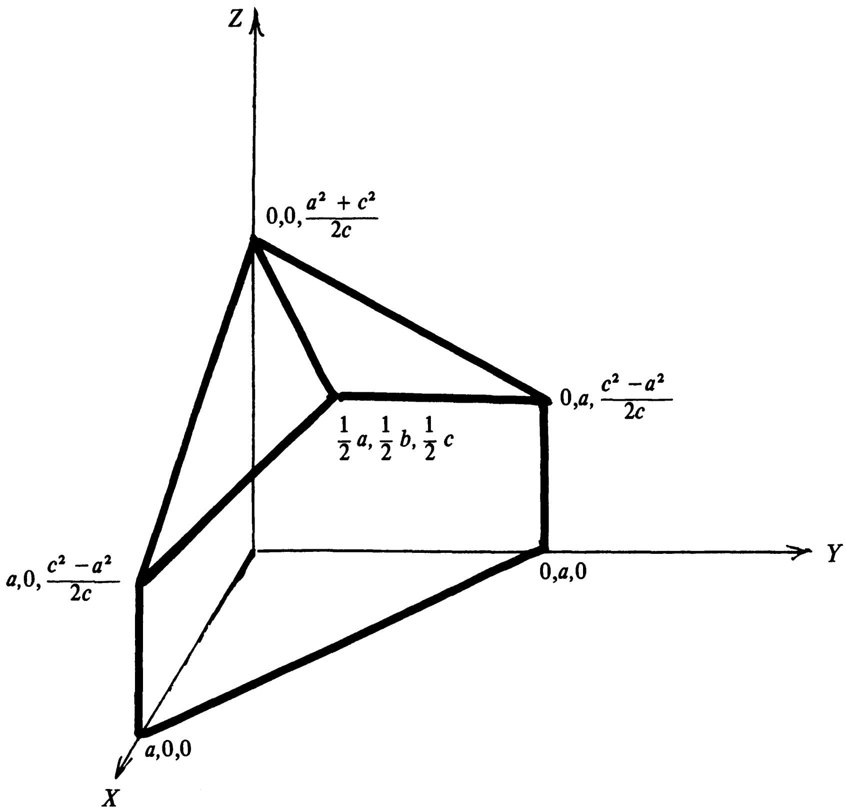
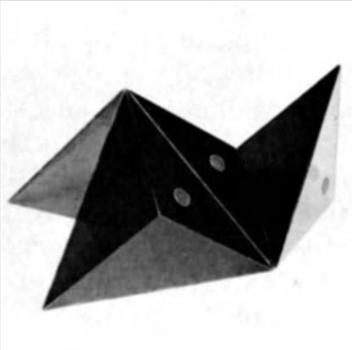


FIGURE 18-6 Dirichlet Domain of face-centered tetragonal lattice

These examples of orthorhombic and tetragonal lattices illustrate the importance of the truncated octahedron, the rhombohedral dodecahedron, and the dodecahedron/prism combination and their transformations into each other.



Coda: Unwrapping the Cube: A Photographic Essay

(With technical assistance of C. Todd Stuart and photography by Bruce Anderson)

In *Synergetics* (R. Buckminster Fuller, E. J. Applewhite, and A. L. Loeb, op. cit.) and in Chapter 6 of this volume we discussed the instability of the cube, and the stability of triangulated solids such as the octahedron and the tetrahedron. The importance of the rhombohedral dodecahedron was stressed in Chapters 16 and 17.

The cube does have some significant properties of its own. It is the only one of the five Platonic solids that is also a space-filler. In a gravitational field that is constant in direction and magnitude, a cube may be placed such that each of its faces is either parallel or perpendicular to the lines of force. Both of these properties have contributed to the use of the cube, or more generally, a rectangular hexahedron, as a dwelling unit. The advantage of a horizontal floor is undeniable, but the necessity for horizontal ceilings and vertical walls may be argued. Walls are not created for the purpose of hanging pictures: art, in point of fact, may have been forced into a two-dimensional framework by the existence of vertical walls.

The author has developed a number of devices for familiarizing his students in Design Science at Harvard with non-cubical forms, and for relating these forms to the familiar cube. Two of these models (constructed by Robert Stortz after a design by the author) transform two cubes into a single rhombohedral structure. In their various configurations these structures present interesting sculptural forms. The transformations were photographed in various intermediate stages, showing the cube unfolding, and finally joining with a partner to generate a rhombohedral dodecahedron. There are two fundamentally different processes by which these transformations may be accomplished; both have been photographed. In addition to their pedagogical value, these two sequences appear to have an esthetic significance of their own; for this reason we present them as a conclusion to "Space Structures: Their Harmony and Counterpoint."

The cube may be subdivided into four mutually congruent portions, named oc-tets*. The first transformation photographed is based on this subdivision. We showed (cf. Table 11-3) that degenerate stellation of a regular tetrahedron yields a cube. A cube thus may contain an inscribed tetrahedron (cf. Figure 17-3); the four polyhedra *inside* the cube but *outside* the inscribed tetrahedron constitute four octants of a regular octahedron. We subdivide the inscribed tetrahedron into four mutually congruent triangular pyramids, each having as base a face of the tetrahedron, and as apex the common center of the cube and the tetrahedron. The *oc-tet* is defined as the unsymmetrical triangular bipyramid formed by placing the octant of the octahedron with its triangular base in contact with the base of the quarter tetrahedron. The oc-tet thus has one apex where the edges meet perpendicularly, whereas at its second apex the edges meet at the angle $(180° - \delta)$, whose cosine equals minus one-third (cf. p. 135-136). The cube thus may be subdivided into four oc-tets whose rectangular apices constitute four vertices of the cube, and whose obtuse apices meet at the cube center. Keeping this subdivision in mind, the photographic sequence proceeds as follows:

***term devised independently by R. Buckminster Fuller (oc-tet truss) and by Janet Varon in a Freshman Seminar term paper (1975)**

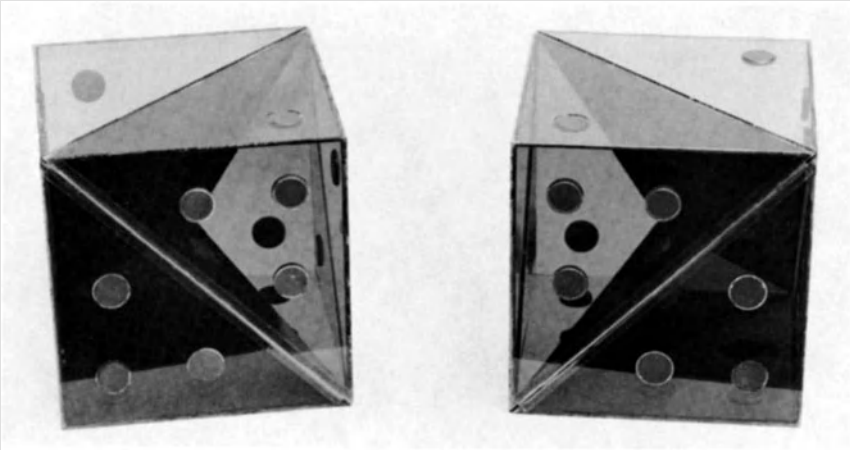


FIGURE 19-1 Two cubes. Each face is bisected by a diagonal: the six diagonals in each cube constitute the edges of the tetrahedron inscribed in the cube.

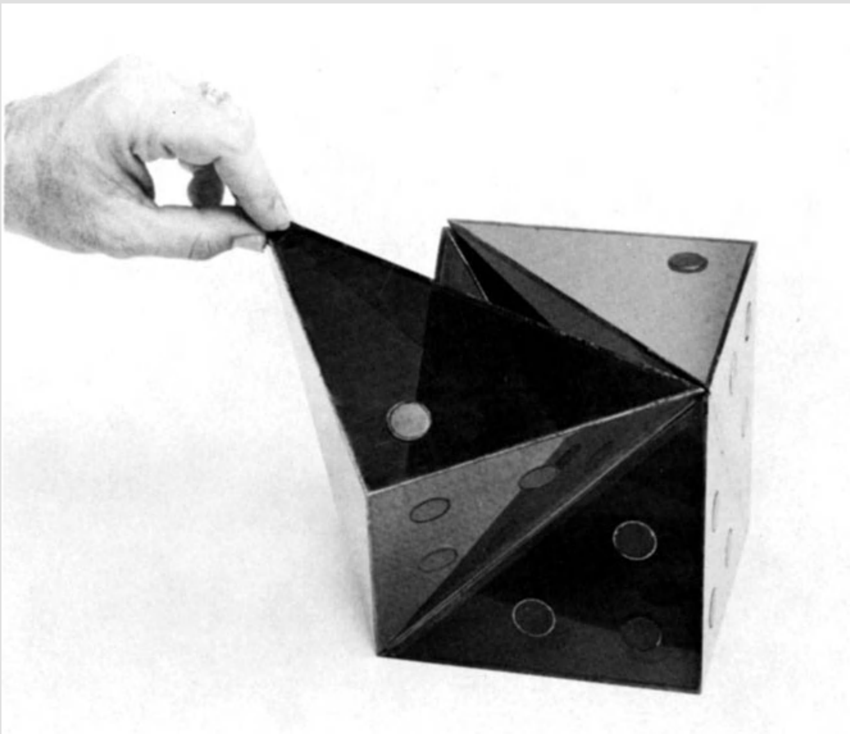


FIGURE 19-2 An oc-tet emerges out of the cube.

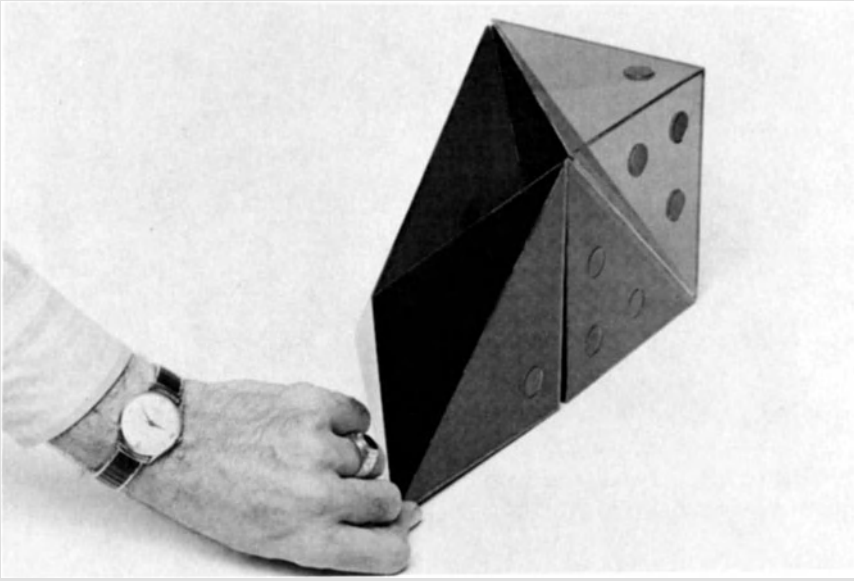


FIGURE 19-3 The oc-tet is folded back along a face of the cube.

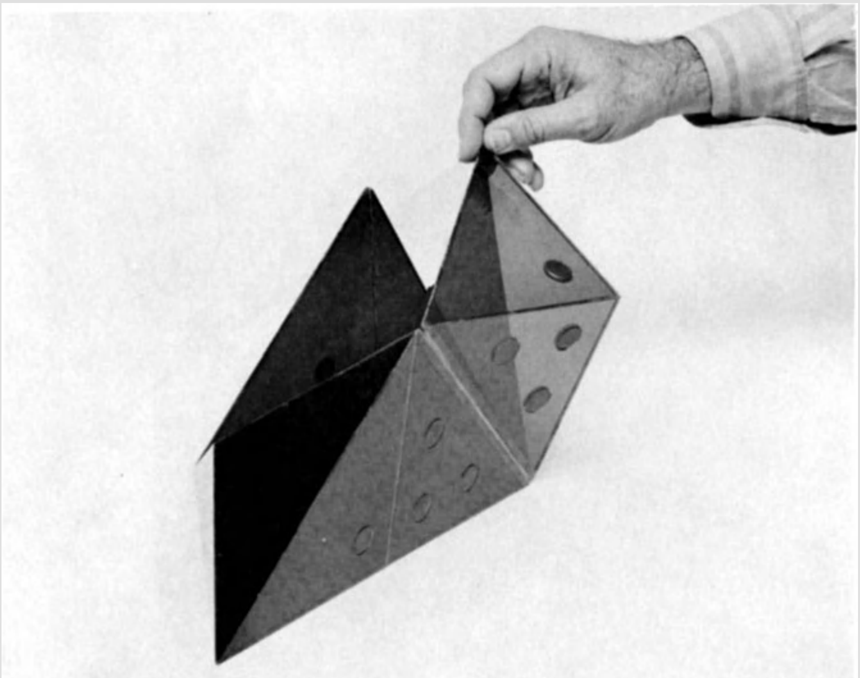
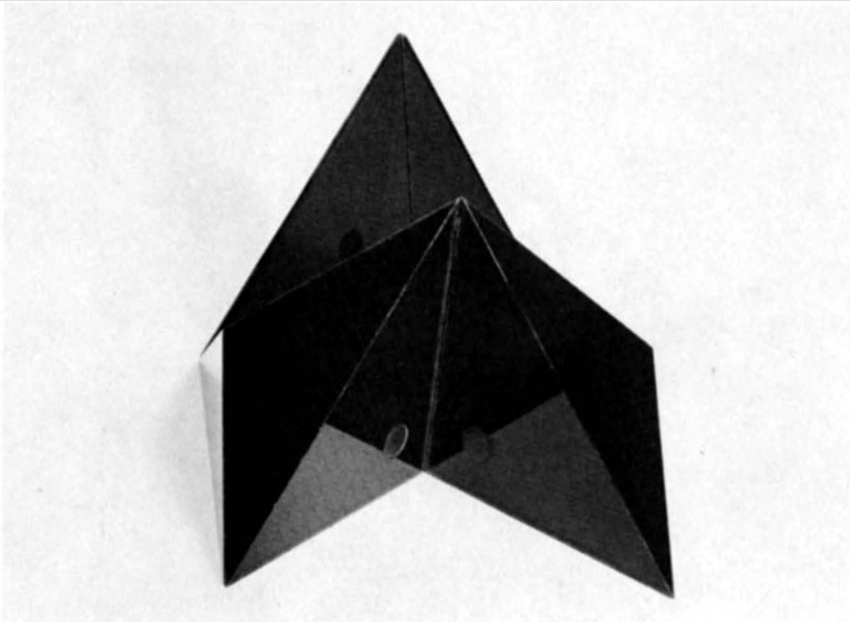
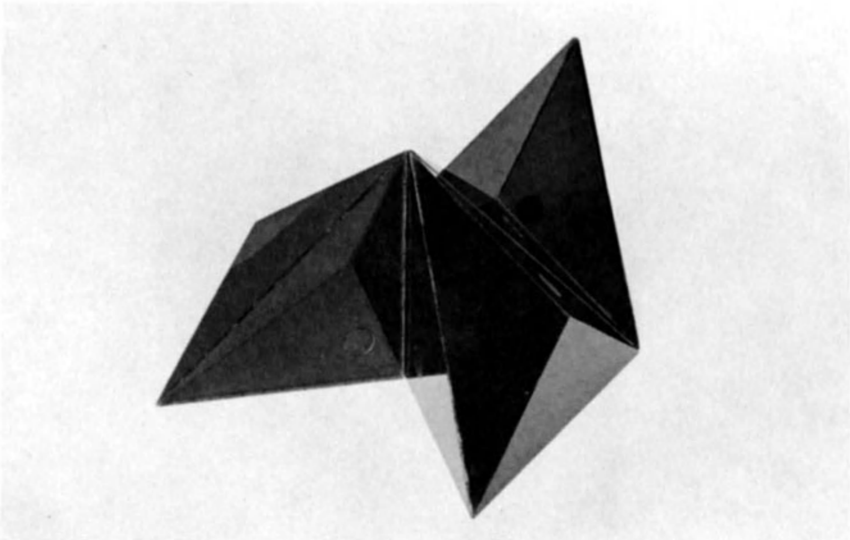
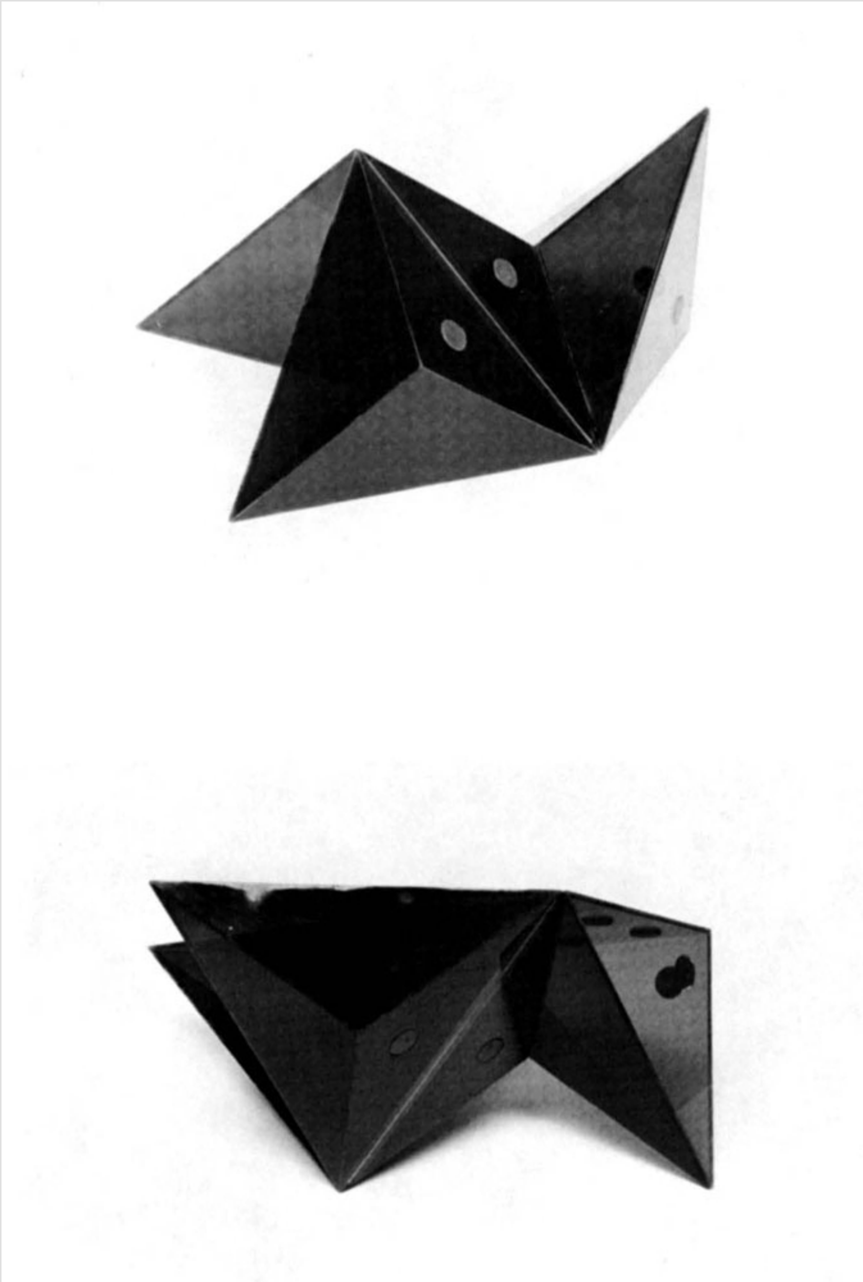


FIGURE 19-4 A second oc-tet emerges and is folded back.

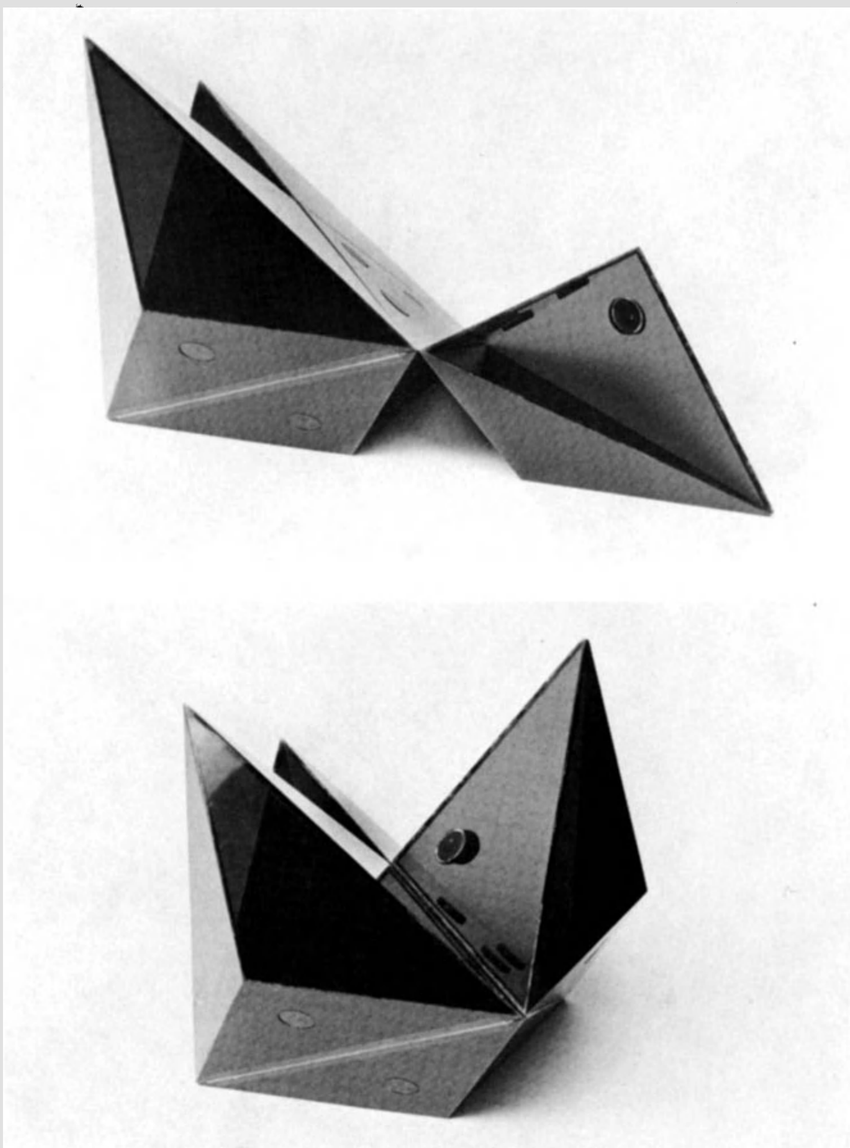




FIGURES 19-5, 19-6, 19-7, and 19-8 The resulting configuration (a design for a church?) is rotated to show different perspectives.



FIGURES 19-7, 19-8 (series cont. from p. 151) The resulting configuration is rotated to show different perspectives.



FIGURES 19-9 and 19-10 A third oc-tet is folded back along a face diagonal of the original cube. The original cube has now been turned inside out, so that the rectangular vertices of the oc-tets, which originally constituted cube vertices, now meet at a common point. The obtuse vertices of the oc-tets, which originally met at the center of the cube, are now turned out.

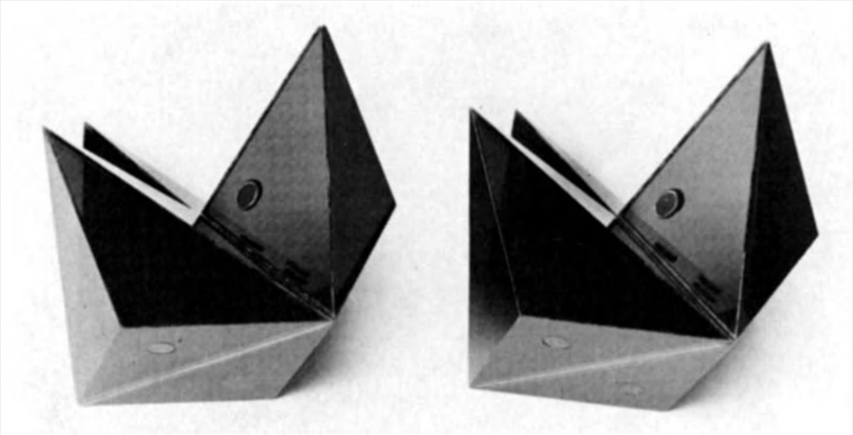


FIGURE 19-11 Meanwhile, the second cube has experienced the same transformation as has its twin.

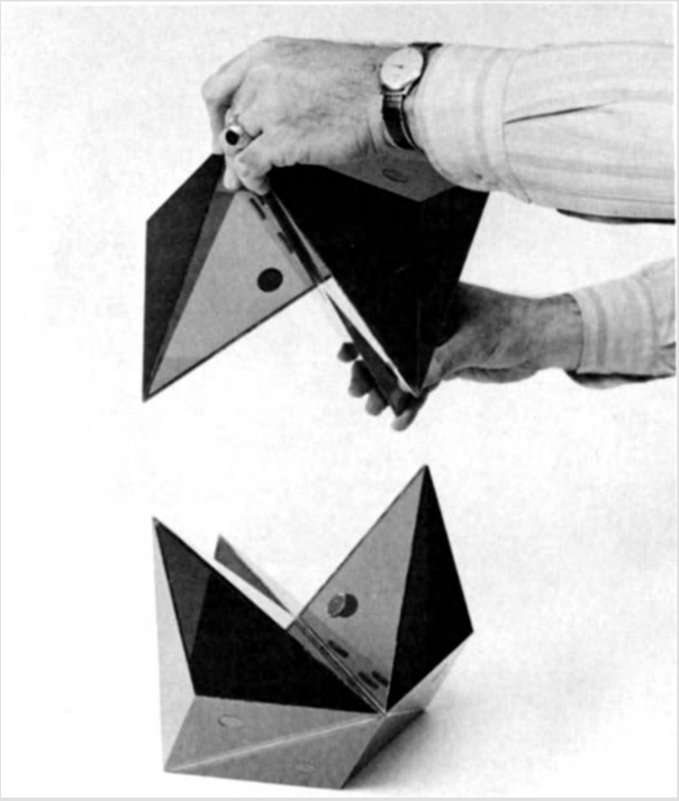
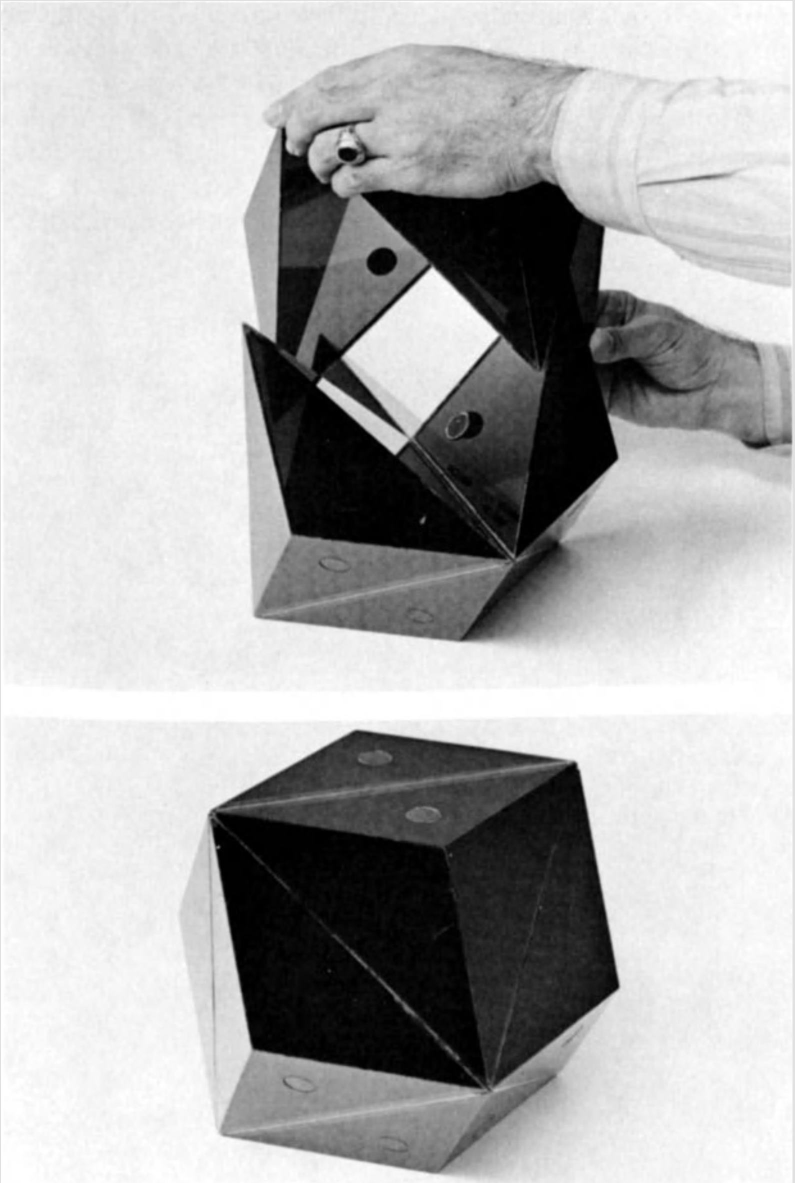


FIGURE 19-12 The two transformed cubes are brought into alignment.



FIGURES 19-13 and 19-14. The two transformed cubes are joined together to form a rhombohedral dodecahedron. The rhombohedral dodecahedron has twelve rhombic faces. One set of surface angles have a cosine equal to plus one-third (the angle δ), the other set are their supplements.

The second transformation sequence is based on subdividing the cube into six square pyramids, each having a face of the cube as its base, and the center of the cube as its apex. Two of such square pyramids, joined at their bases, constitute the space-filling octahedron discussed on pp. 133-137, the Dirichlet Domain of the J-complex. The four edges meeting at the apices, it is recalled, make angles and $(180° - \delta)$ with each other. The second transformation sequence unfolds as follows:

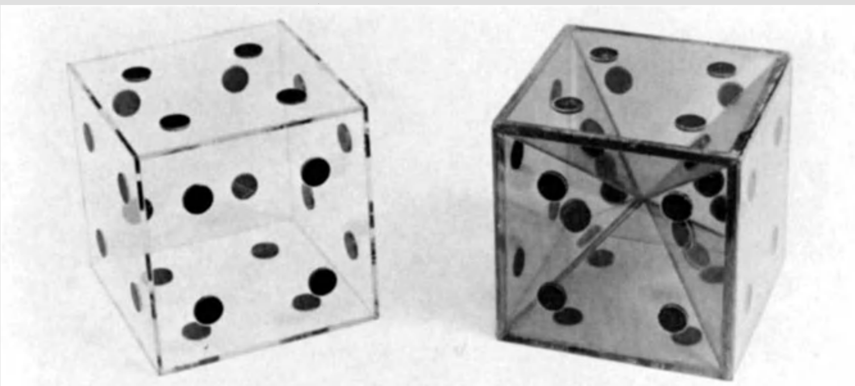
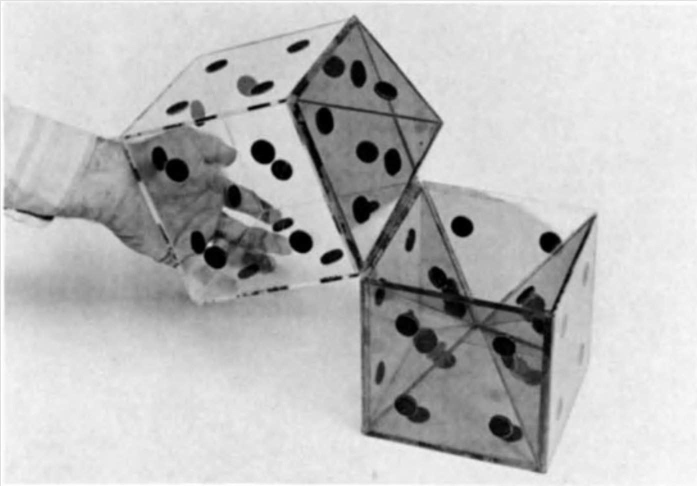
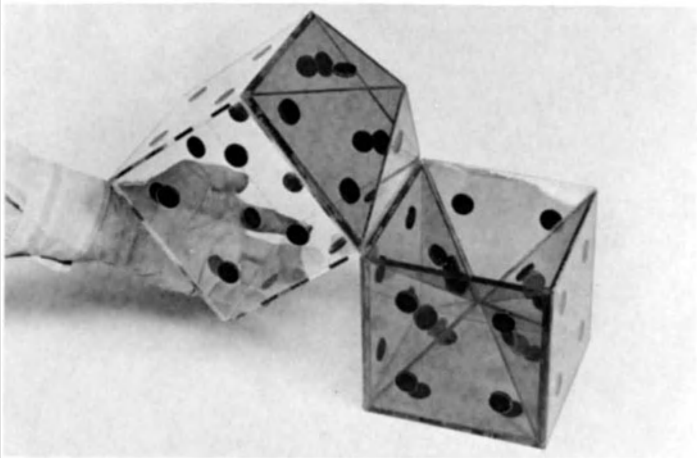
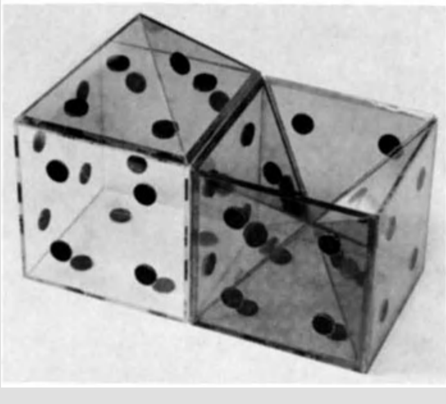


FIGURE 19-15 Two cubes. One of the cubes is subdivided into six pyramids, the other is not. (The discs are magnets, which hold the pyramids together where this is desired.)

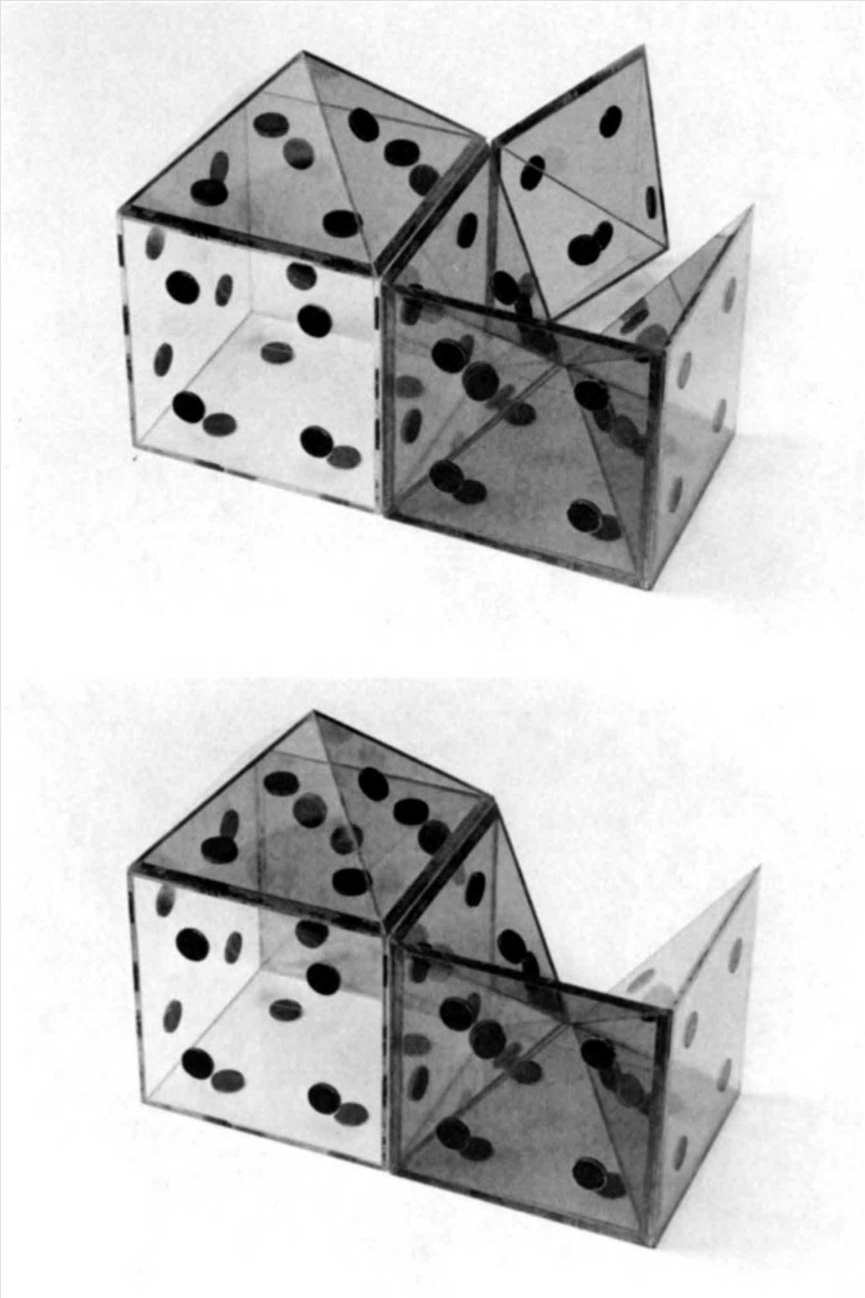
FIGURE 19-16 The cubes are juxtaposed with one face contiguous.







FIGURES 19-17, 19-18 and 19-19 A pyramid is transferred from one cube to the other.



FIGURES 19-20 and 19-21 A second pyramid is removed from the first cube and transferred to the second cube.

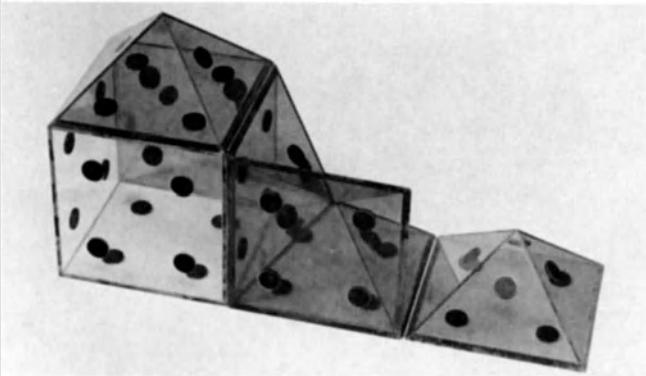
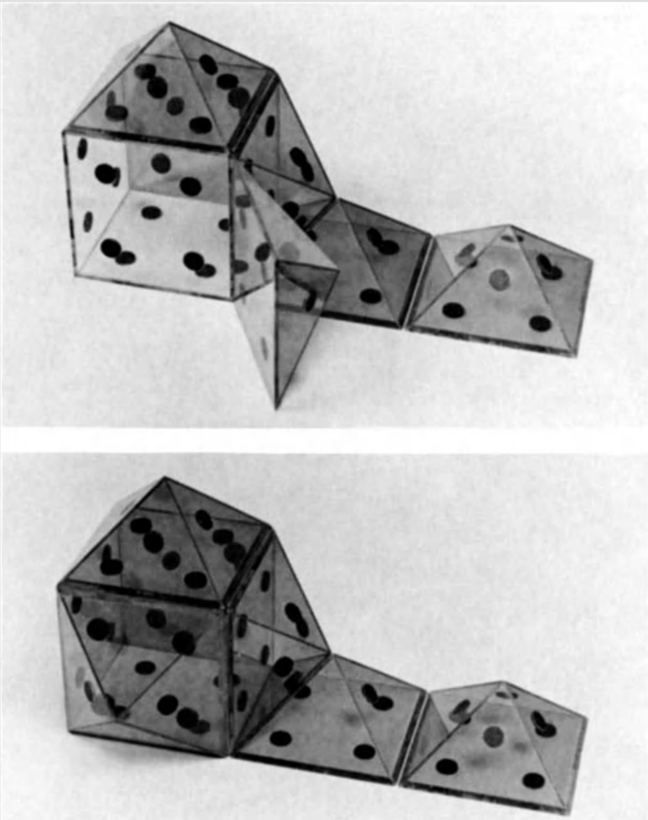
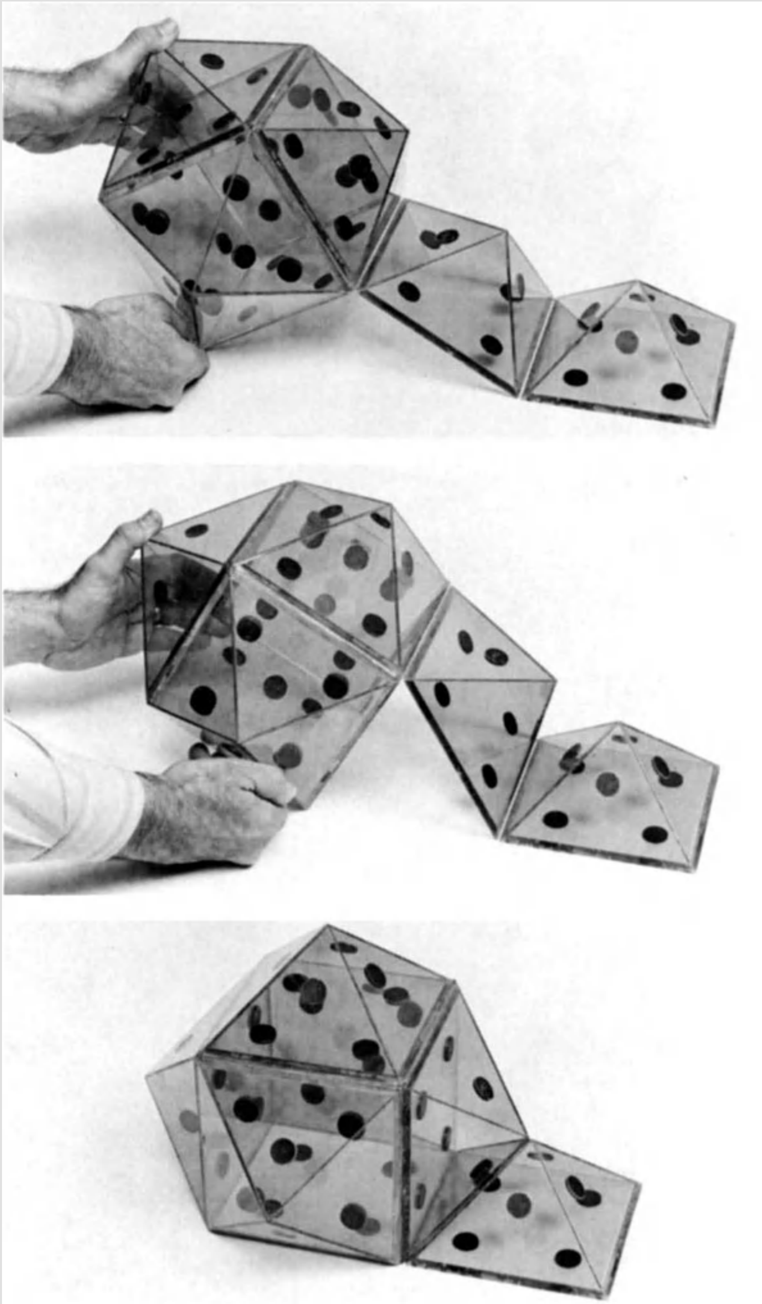


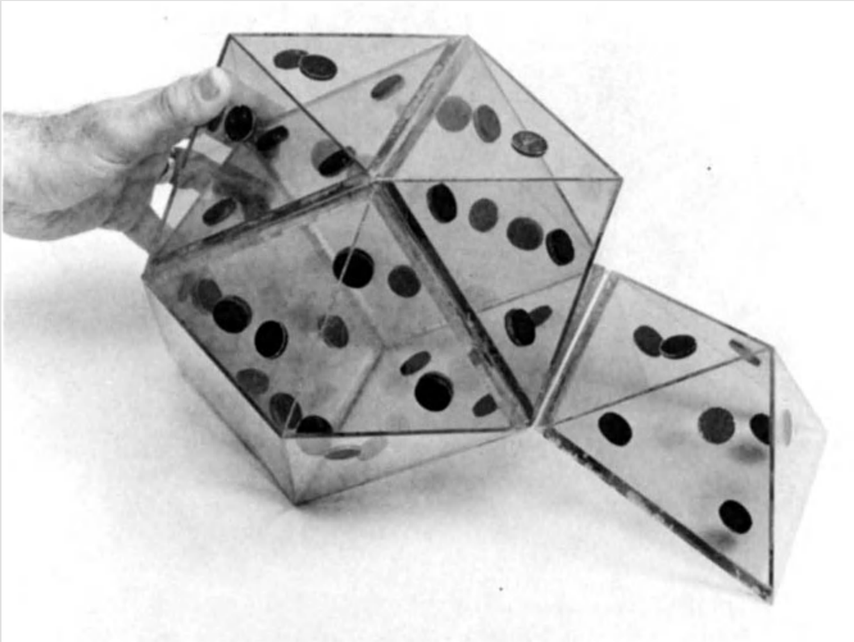
FIGURE 19-22 A third pyramid is unfolded from the first cube, and left lying beside it.

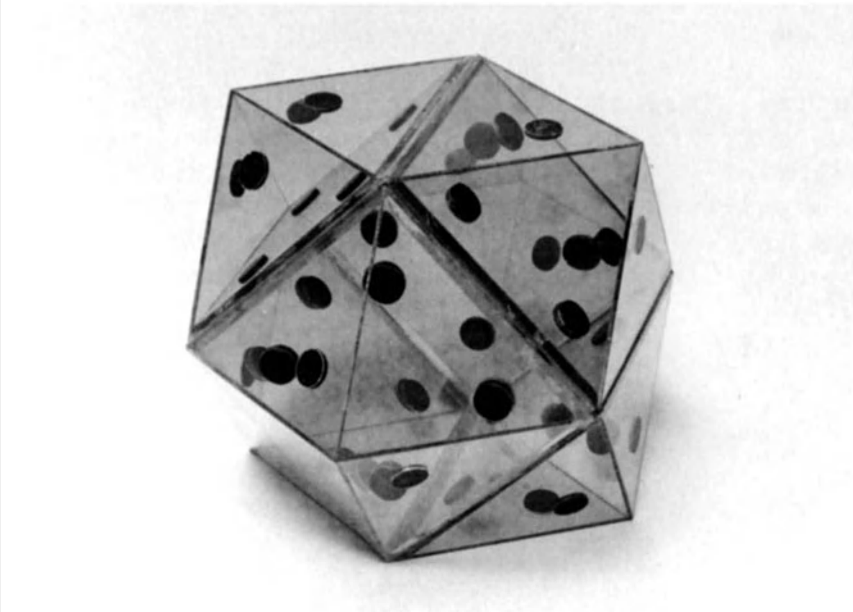


FIGURES 19-23 and 19-24 A fourth pyramid is transferred. Of the original first cube only two pyramids remain.



FIGURES 19-25, 19-26 and 19-27 One of the remaining pyramids is annexed by the second cube.





FIGURES 19-28 and 19-29 The sixth pyramid is annexed, completing the formation of the rhombohedral dodecahedron.

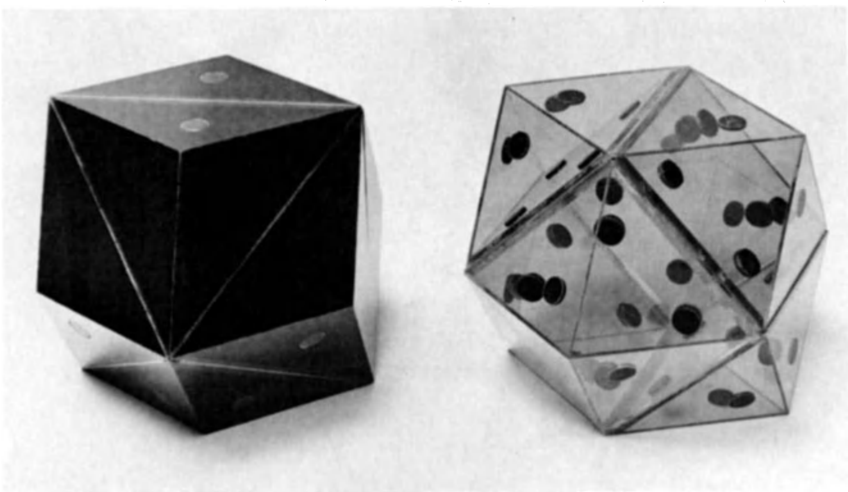


FIGURE 19-30 Both transformations of the cube, although fundamentally different processes, result in the same final form.

Note that of the face diagonals of the resulting rhombohedral dodecahedron the long ones constitute edges of an octahedron, the short ones the edges of a cube: the dodecahedron is a degenerate stellation of either the octahedron or the cube.

Corollary: This photographic essay demonstrates the importance in space structures of the angle δ, whose cosine equals *one-third.* This angle is a surface angle of the rhombohedral dodecahedron *as well as* the angle of intersection between body diagonals of the cube and the angle subtended at the center of a regular tetrahedron by any pair of its vertices. These transformations demonstrate the reason for this "as well as."

Bibliography

A great deal has been published since the first appearance of *Space Structures*. William L. Hall, Jack C. Gray and the author have maintained a reading shelf and a file of reprints, which have been the major source of this appended bibliography on Design Science. As this bibliography transcends the boundaries of many disciplines, it has not been an easy task to keep abreast of developments. It has been the author's experience that a well-informed grapevine can be more effective than a computer search and therefore wishes to express his thanks to the many friends who sent in reprints and references. However, the author wishes to apologize for errors of commission and of omission which appear to be virtually inevitable.

BOOKS:

Abbott, A.: *Flatland,* Dover, New York (1952, 1983).

Albairn, K., J. Miall Smith, S. Steele and D. Walker: *The Language of Pattern,* Thames & Hudson, London (1974).

Abelson, H. and A. diSessa: *Turtle Geometry,* MIT Press, Cambridge (1981).

Adams, James L.: *Conceptual Blockbusting,* W. H. Freeman & Co., New York (1974).

Alexander, Christopher: *Notes on the Synthesis of Form,* Harvard Univ. Press, Cambridge (1964).

Alexander, Christopher et al.: *A Pattern Language,* Oxford Univ. Press, New York (1977).

Applewhite, E. J.: *Cosmic Fishing,* Macmillan, New York (1977).

Applewhite, E. J., ed.: *Synergetics Dictionary, the Mind of Buckminster Fuller,* Garland, New York (1986).

Arnheim, Rudolf: *Visual Thinking,* Univ. of California Press, Berkeley (1969).

Arnstein, Bennett: *Origami Polyhedra,* Exposition Press, New York (1968).

Bacon, G. E.: *The Architecture of Solids,* Taylor and Francis, London (1981).

Baglivo, Jenny, and Jack A. Graver: *Incidence and Symmetry in Design and Architecture,* Cambridge Univ. Press, Cambridge, U.K. (1983).

Ball, W. W. R., and H. S. M. Coxeter: *Mathematical Recreations and Essays,* Dover, New York (1986).

Banchoff, Thomas, and John Weber: *Linear Algebra through Geometry,* Springer, New York (1983).

Banchoff, Thomas: *Behind the Third Dimension,* Scientific American Library, W. H. Freeman, New York (1990).

Barratt, Krome: *Logic and Design, the Syntax of Art, Science and Mathematics,* Eastview, Westfield, N.J. (1980).

Beard, Col. R. S.: *Patterns in Space,* Creative Publications, Palo Alto (1973).

Behnke, H., F. Bachman, K. Fladt and H. Kunle, eds.; S. H. Gould, transl.: *Fundamentals of Mathematics,* Volume II, MIT Press, Cambridge (1974).

Belmont, Joseph: *L'Architecture Création Collective,* Les Editions Ouvrières, Paris (1970).

Belov, N. V.: *Derivation of the 230 Space Groups,* Leeds Philosophical and Literary Society, Leeds (1957).

Bernal, Ivan, Walter C. Hamilton and John C. Ricci: *Symmetry: A Stereoscopic Guide for Chemists,* W. H. Freeman & Co., New York (1972).

Blackwell, William: *Geometry in Architecture,* Wiley, New York (1984).

Boles, Martha, and Rochelle Newman: *The Golden Relationship: Art, Math, Nature, I,* Pythagorean Press, Bradford, Mass. (1983).

Bool, Flip, ed.: *M. C. Escher (1898–1972),* Haags Gemeentemuseun (1986).

Borrego, John: *Space Grid Structures,* MIT Press, Cambridge (1968).

Boys, C. V.: *Soap Bubbles and the Forces which Mould Them,* Doubleday-Anchor, Garden City, New York (1959).
Bragdon, Claude: *A Primer of Higher Space,* Omen Press (1972).
Bragdon, Claude: *Projective Ornament,* Omen Press.
Bragg, Sir Lawrence: *Crystal Structures of Minerals,* Cornell Univ. Press, Ithaca, N.Y. (1965).
Bragg, W. L.: *The Crystalline State,* Macmillan, New York (1934).
Bravais, M. A., Amos J. Shalar, transl.: *On the Systems Formed by Points Regularly Distributed on a Plane or in Space,* Crystallographic Society of America (1949).
Brenneman, Richard J.: *Fuller's Earth: A Day with Bucky and the Kids,* St. Martin, New York (1984).
Brisson, David, ed.: *Hypergraphics,* AAAS Selected Symposium Series, Westview Press, Boulder (1978).
Buerger, Martin: *Elementary Crystallography,* Wiley, New York (1963).
Buerger, Martin: *Introduction to Crystal Geometry,* McGraw Hill, New York (1971).
Burckhardt, J. J.: *Die Bewegungsgruppen der Kristallographie,* Birkhäuser, Basel (1966).
Burns, Gerald, and A. M. Glazer: *Space Groups for Solid State Scientists,* Academic Press, New York (1978).
Burt, Michael: *Spatial Arrangement and Polyhedra with Curved Surfaces and their Architectural Applications,* Technion, Haifa (1966).
Clark, Graham M.: *The Structures of Non-Molecular Solids,* Wiley, New York (1972).
Coffin, Stewart: *The Puzzling World of Polyhedral Dissections,* Oxford Univ. Press, New York (1990).
Collier, Graham: *Form, Space and Vision,* Prentice Hall, Englewood Cliffs, N.J. (1963).
Cotton, F. Albert: *Chemical Applications of Group Theory,* Wiley, New York (1963).
Coxeter, H. S. M.: *Introduction to Geometry,* Wiley, New York (1961).
Coxeter, H. S. M. and W. O. Moser: *Generators and Relations for Discrete Groups,* Springer, New York (1965).
Coxeter, H. S. M., and S. L. Greitzer: *Geometry Revisited,* Mathematical Association of America, Washington, D.C. (1967).

Coxeter, H. S. M.: *Twelve Geometric Exercises,* Southern Illinois Univ. Press, Carbondale (1968).
Coxeter, H. S. M.: *Regular Polytopes,* Methuen, London (1948), Dover, New York (1973).
Coxeter, H. S. M.: *Complex Polytopes,* Cambridge Univ. Press (1975).
Coxeter, H. S. M., P. Du Val, H. T. Flather and J. F. Petrie: *The Fifty-nine Icosahedra,* Springer, New York (1982).
Coxeter, H. S. M., M. Emmer, R. Penrose and M. L. Teuber, eds.: *M. C. Escher, Art and Science,* Elsevier Science Publishers, Amsterdam (1986).
Critchlow, Keith: *Order in Space, A Design Source Book,* Viking Press, New York (1970).
Critchlow, Keith: *Islamic Patterns,* Schocken, New York (1976).
Dale, J. E. and F. L. Milthorpe, eds.: *The Growth and Functioning of Leaves,* Cambridge Univ. Press, New York (1982).
Dimond, S. J. and D. A. Blizard, eds.: *Evolution and Lateralization of the Brain,* New York Academy of Sciences (1977).
Doczi, Gyorgy: *The Power of Limits,* Shambala, Boulder (1981).
Doeringer, Suzannah et alia, eds.: *Art and Technology,* MIT Press, Cambridge (1970).
Doxiadis, C. A.: *Architectural Space in Ancient Greece,* MIT Press, Cambridge (1972).
Drabkin, David: *Nature's Architecture,* Univ. of Pennsylvania Press, Philadelphia (1975).
Drew, Philip: *Frei Otto-Form and Structure,* Westview Press, Boulder (1976).
Edgerton, S. Y., Jr.: *The Renaissance Rediscovery of Linear Perspective,* Basic Books (1975).
Edmondson, Amy: *A Fuller Explanation: The Synergetic Geometry of R. Buckminster Fuller,* Birkhäuser Design Science Collection, A. L. Loeb, ed. (1986).
Edwards, Betty: *Drawing on the Right Side of the Brain,* J. P. Tarcher, Los Angeles (1979).
Edwards, Edward B.: *Patterns and Designs with Dynamic Symmetry,* Dover, New York (1967).
Ehrenfeucht, Aniela: *The Cube made interesting,* Pergamon, New York (1964).
Eisenhart, L. P.: *A Treatise on the Differential Geometry of Curves and Surfaces,* Dover, New York (1960).

El-Said, Issam, and Ayse Parman: *Geometric Concepts in Islamic Art,* World of Islam, London (1976).
Emerton, Norma: *The Scientific Reinterpretation of Form,* Cornell Univ. Press, Ithaca (1984).
Engel, Peter: *Geometric Crystallography,* Reidel, Boston (1986).
Engel, Peter: *Folding the Universe,* Vintage, New York (1988).

[Note: These latter two references are by two different, identically named authors. The former is Swiss, the latter American.]

Ernst, Bruno: *The Magic Mirror of M. C. Escher,* Random House, New York (1976).
Escher, M. C.: *The Graphic Work,* Meredith, New York (1967), and various other publishers.
Escher, M. C.: *Escher on Escher: Exploring the Infinite,* Abrams, New York (1989).
Falicov, L. M.: *Group Theory and its Applications,* Univ. of Chicago Press, Chicago (1966).
Fedorov, E. S., D. and K. Harker, transl.: *Symmetry of Crystals,* American Crystallographic Association (1971).
Fejes Toth, L.: *Regular Figures,* Macmillan, New York (1964).
Flegg, H. Graham: *From Geometry to Topology,* Crane Russak and Co., Inc., New York (1974).
Fleury, Michael: *Graphisme et Géométrie,* Presses de l'Univ. de Québec, Québec (1986).
Fischer, W., H. Burzlaff, E. Hellner and J. D. H. Donnay: *Space Groups and Lattice Complexes,* U.S. Department of Commerce, National Bureau of Standards (1973).
Francis, George K.: *A Topological Picture Book,* Springer, New York (1987).
Fuller, R. Buckminster: *No More Second-hand God,* Southern Illinois Univ. Press, Carbondale (1963).
Fuller, R. Buckminster: *Intuition,* Doubleday, New York (1972).
Fuller, R. Buckminster: *Nine Chains to the Moon,* Southern Illinois Univ. Press, Carbondale (1963).
Fuller, R. Buckminster: *Operating Manual for Spaceship Earth,* Southern Illinois Univ. Press, Carbondale (1969).
Fuller, R. Buckminster, with E. J. Applewhite and A. L. Loeb: *Synergetics,* Macmillan, New York (1975).
Fuller, R. Buckminster, with E. J. Applewhite, ed.: *Synergetics 2,* Macmillan, New York (1979).

Fuller, R. Buckminster: *Critical Path,* St. Martin's Press, New York (1981).

Fuller, R. Buckminster: *Grunch of Giants,* St. Martin's Press, New York (1983).

Gabel, Medard: *Energy, Earth and Everyone,* Straight Arrow Books, San Francisco (1975).

Gardner, Martin: *Logic Machines and Diagrams,* New York (1958).

Gardner, Martin: *Mathematical Puzzles and Diversions,* Simon & Schuster, New York (Vol. 1, 1959; Vol. 2, 1961).

Gardner, Martin: *The Ambidextrous Universe,* Scribner, New York (1979).

Gardner, Martin: *Penrose Tiles to Trapdoor Codes,* W. H. Freeman, New York (1988).

Gasson, Peter C.: *Geometry of Spatial Forms,* Wiley, New York (1983).

Ghyka, Matila: *The Geometry of Art and Life,* Sheed and Ward (1946), reprint Dover, New York (1977).

Ghyka, Matila: *Geometrical Composition and Design,* Alex Tiranti (1964).

Gillespie, Ronald J. *Molecular Geometry,* Van Nostrand Reinhold, London (1972).

Gips, James: *Shape Grammars and their Uses,* Birkhäuser, Boston (1975).

Gombrich, E. H.: *Art and Illusion,* Princeton Univ. Press, Princeton (1960).

Gombrich, E. H.: *The Sense of Order,* Cornell Univ. Press, Ithaca (1984).

Gomez-Alberto, Perez: *Architecture and the crisis of Modern Science,* MIT Press, Cambridge (1984).

Gould, Stephen Jay: *Ontogeny and Philogeny,* Harvard Univ. Press, Cambridge (1977).

Graziotti, Ugo Adriano: *Polyhedra: The Realm of Geometric Beauty,* San Francisco (1962).

Gregory, R. L.: *The Intelligent Eye,* McGraw Hill, New York (1970).

Grenander, U.: *Pattern Synthesis,* Springer, New York (1976).

Grünbaum, Branko: *Convex Polytopes,* Interscience, New York (1967).

Grünbaum, Branko and G. C. Shephard: *Tilings and Patterns,* Freeman, New York (1987).

Gullette, Margaret M.: *The Art of Teaching,* Alpine, Stoughton, Mass. (1982).

Hadanard, Jacques: *The Psychology of Invention in the Mathematical Field,* Dover, New York (1954).

Hahn, Theo, ed.: *International Tables for Crystallography,* Reidel, Boston (1985 & '88).

Halmos, Paul R.: *Naive Set Theory,* Van Nostrand, New York (1960).

Harary, Frank, Robert Z. Norman, and Dorwin Cartwright: *Structural Models: An Introduction to the Theory of Directed Graphs,* Wiley, New York (1965).

Harary, Frank, ed.: *Topics in Graph Theory,* N.Y. Acad. of Sciences, New York (1979).

Hargittai, Istvan, ed.: *Symmetry, Unifying Human Understanding,* Pergamon, New York (Vol. 1, 1986 & Vol. 2, 1989).

Hargittai, I. and M.: *Symmetry through the eye of a Chemist,* VCH Publishers, New York (1987).

Harlan, Calvin: *Vision and Invention,* Prentice Hall, Englewood Cliffs, N.J. (1986).

Hecker, Zvi: *Polyhedral Architecture,* The Israel Museum (1976).

Heidegger, Martin: *On the Way to Language,* Harper and Row, New York (1971).

Henderson, Linda: *The Fourth Dimension and Non-Euclidean Geometry in Modern Art,* Princeton Univ. Press, Princeton (1983).

Herzog, Thomas: *Pneumatic Structures,* Oxford Univ. Press, New York (1976).

Hilbert, David, and S. Cohn-Vossen: *Geometry and the Imagination,* Chelsea, New York (1952).

Hildebrandt, Stefan, and Anthony Tromba: *Mathematics and Optimal Form,* Freeman, New York (1985).

Hill, Anthony: *DATA: Directions in Art, Theory and Aesthetics,* Farber and Farber (1968).

Hilliard, Garland K., ed.: *Proceedings of the International Conference on Descriptive Geometry,* Wm. Brown, Dubuque, Iowa (1978).

Hilton, Harold: *Mathematical Crystallography,* Dover, New York (1963).

Hilton, Peter and Jean Pederson: *Build your own Polyhedra,* Addison-Wesley, Reading, Mass. (1988).

Hoag, John: *Islamic Architecture,* Rizzoli, New York (1987).

Hofstadter, Douglas R.: *Gödel, Escher, Bach,* Basic Books, New York (1979).

Holden, Alan and Phylis Singer: *Crystals and Crystal Growing,* Anchor, Garden City (1960).

Holden, Alan: *The Nature of Solids,* Columbia Univ. Press, New York (1965).

Holden, Alan: *Shapes, Space and Symmetry,* Columbia Univ. Press, New York (1971).

Holden, Alan: *Bonds between Atoms,* Oxford Univ. Press, New York (1971).

Holden, Alan: *Orderly Tangles,* Columbia Univ. Press (1983).

Huntley, H. E.: *The Divine Proportion,* Dover, New York (1970).

International Tables for X-ray Crystallography, Reidel, Dordrecht/Boston (1983 ff).

Ivins, William M., Jr.: *Art and Geometry,* Harvard Univ. Press, Cambridge (1946) & Dover, New York (1969).

Jaswon, M. A.: *An Introduction to Mathematical Crystallography,* American Elsevier, New York (1965).

Judson, Horace F.: *The Search for Solutions,* Holt, Rinehart & Winston, New York (1980).

Kappraff, Jay: *Connections, the Geometric Bridge between Art and Science,* McGraw Hill, New York (1990).

Kenner, Hugh: *Bucky, A Guided Tour of Buckminster Fuller,* Wm. Morrow (1973).

Kenner, Hugh: *Geodesic Math and How to Use it,* Univ. of California Press, Berkeley (1976).

Kepes, G., ed.: *The Module, Proportion, Symmetry and Rhythm,* Braziller, New York (1965).

Kepler, Johann: *The Six-Cornered Snowflake,* Oxford Univ. Press, London (1966).

Kitaigorodskiy, A. I.: *Order and Disorder in the World of Atoms,* Springer, New York (1967).

Klein, Felix: *Lectures on the Icosahedron and the Solution of Equations of the Fifth Degree,* Dover, New York (1956).

Kline, Morris: *Mathematics in Western Culture,* Oxford Univ. Press, New York (1953).

Kline, Morris: *Mathematical Thought from Ancient to Modern Times,* Oxford Univ. Press, New York (1972).

Kline, Morris: *Mathematics and the Physical World,* Dover, New York (1981).

Lakatos, Imre: *Proofs and Refutations,* Cambridge Univ. Press, New York (1976).

Lalvani, Haresh: *Structures and Hyperstructures,* Pratt Institute, Private Publication, Brooklyn (1977).

Lalvani, Haresh: *Transpolyhedra,* Pratt Institute, Private Publication, Brooklyn (1977).

Lauwerier, Hans: *Fractals* (in Dutch), Aramith, Amsterdam (1987).

Lauwerier, Hans: *Symmetrie-Regelmatige Strukturen in de Kunst* (in Dutch), Aramith, Amsterdam (1988).

Lawler, Robert: *Sacred Geometry: Philosophy and Practice,* Thames and Hudson, New York (1982).

Laycock, Mary: *Bucky for Beginners,* Activity Resources, Hayward, Calif. (1984).

LeCorbeiller, Philippe, ed.: *The Language of Science,* Basic Books, New York (1963).

LeCorbeiller, Philippe: *Dimensional Analysis,* Appleton Century Crofts, New York (1966).

Leeman, Fred: *Hidden Images,* Abrams, New York (1975).

Lindgren, H.: *Geometric Dissections,* Dover, New York (1972).

Locher, J. L., ed.: *De Werelden van M. C. Escher,* Meulenhoff, Amsterdam (1971), translated as *The World of M. C. Escher,* Abradale, Steven Sterk, Utrecht (1988).

Locher, J. L.: *Escher,* Thames and Hudson, London (1982).

Loeb, A. L.: *Color and Symmetry,* Wiley, New York (1971), Krieger, New York (1978).

Loeb, A. L.: *Crimson Heather, Twenty-one Scottish Country Dances,* Boston Branch Book Store, Royal Scottish Country Dance Society, Boston (1986).

Loeb, A. L.: *Space Structures, their Harmony and Counterpoint,* Addison-Wesley, Reading, Mass. (1976).

Lord, Athena: *Pilot for Spaceship Earth,* MacMillan, New York (1978).

Lysternik, L. A.: *Convex Figures and Polyhedra,* Dover, New York (1963).

Malina, Frank J., ed.: *Visual Art, Mathematics and Computers,* Pergamon, New York (1979).

Mandelbrot, Benoit B.: *The Fractal Geometry of Nature,* Freeman, New York (1983).

Maor, Eli: *To Infinity and Beyond,* Birkhäuser, Boston (1987).

March, Lionel, and Philip Stedman: *The Geometry of Environment,* *an Introduction to Spatial Organization in Design,* MIT Press, Cambridge (1978).

Marks, R.: *The Dymaxion World of R. Buckminster Fuller,* Reinhold, Princeton (1960), Doubleday, New York (1973).

Martin, George E.: *Transformational Geometry: An Introduction to Symmetry,* Springer, New York (1982).

Martin, Kenneth: *Chance and Order,* Hillingdon Press, Uxbridge, U.K. (1973).

Martinez, Benjamin, and Jacqueline Block: *Visual Forces,* Prentice Hall, Englewood Cliffs, N.J. (1988).

McHale, John R.: *Buckminster Fuller,* Braziller, New York (1962).

McMullen, P., and G. C. Shephard: *Convex Polytopes and the Upper Bound Conjecture,* Mathematics Society Lecture Note Series, no. 3, Cambridge Univ. Press, New York (1971).

McWeeny, R.: *Symmetry—An Introduction to Group Theory,* Macmillan, New York (1963).

Miller, Arthur: *Imagery in Scientific Thought,* Birkhäuser, Boston (1984).

Minsky, M., and S. Papert: *Perceptrons,* MIT Press, Cambridge (1972 & 1988).

Mitchell, William J.: *Computer Aided Design,* Van Nostrand Reinhold, New York (1977).

Miyazaki, Koji: *Form and Space, Polygons, Polyhedra and Polytopes,* Wiley, New York (1986).

Müller, Edith: *Gruppentheoretische und Strukturanalytische Untersuchungen der Maurischen Ornamente aus der Alhambra,* Rüschlikon, Switzerland (1944).

Nagel, Ernst, and James R. Newman: *Gödel's Proof,* New York Univ. Press, New York (1958).

Needham, Joseph: *Order and Life,* MIT Press, Cambridge (1968).

Nerlick, Graham: *The Shape of Space,* Cambridge Univ. Press, New York (1976).

O'Daffer, P. G.: *Geometry: An Investigative Approach,* Addison-Wesley, Reading, Mass. (1976).

Otto, Frei: *Architecture et Bionique: Constructions Naturelles,* Editions Delta and Spes, Denges, Switzerland (1985).

Parthé, Erwin: *Crystal Chemistry of Tetrahedral Structures,* Gordon and Breach, New York (1964).

Pauling, Linus, and Roger Hayward: *The Architecture of Molecules,* Freeman, New York (1964).

Pearce, Peter, and Susan Pearce: *Polyhedra Primer,* Van Nostrand Reinhold, New York (1978).

Pearce, Peter: *Structure in Nature is a Strategy for Design,* MIT Press, Cambridge (1978).

Pearce, Peter, and Susan Pearce: *Experiments in Form,* Van Nostrand Reinhold, New York (1980).

Pedersen, Jean, and A. Kent: *Geometric Playthings,* Price, Stern and Stone, Los Angeles.

Pedoe, Dan: *Geometry and the Visual Arts,* Dover, New York (1976).

Phillips, F. C.: *An Introduction to Crystallography,* Wiley, New York (1962).

Polya, G.: *How to solve it,* Anchor Books, Garden City, N.Y. (1957).

Preparata, F. P., and M. I. Shamos: *Computational Geometry,* Springer, New York (1985).

Prince, E.: *Mathematical Techniques in Crystallography and Materials Science,* Springer, New York (1982).

Pugh, Anthony: *Polyhedra, A Visual Approach,* Univ. of California Press, Berkeley (1976).

Pugh, Anthony: *An Introduction to Tensegrity,* Univ. of California Press, Berkeley (1976).

Rawson, Philip: *Design,* Prentice Hall, Englewood Cliffs, N.J. (1987).

Reichardt, Jasia: *Cybernetic Serendipity,* Studio International, New York (1968).

Rhodin, Johannes A. G.: *An Atlas of Ultrastructure,* W. B. Saunders, Philadelphia (1963).

Rosen, Joe: *Symmetry Discovered,* Cambridge Univ. Press, New York (1975).

Row, T. Sundra: *Geometric Exercises in Paper Folding,* Dover (reprint of 1905 original), New York (1987).

Rucker, Rudy: *Infinity and the Mind,* Birkhäuser, Boston (1982).

Rucker, Rudy: *The Fourth Dimension: Toward a Geometry of Higher Reality,* Houghton Mifflin, Boston (1984).

Ryschkewitsch, G.: *Chemical Bonding and the Geometry of Molecules,* Van Nostrand Reinhold, New York (1963).

Safdie, Moshe: *For Everyone a Garden,* MIT Press, Cambridge (1974).

Salvadori, Mario: *Why Buildings Stand Up,* McGraw Hill, New York (1969).

Sands, Donald E.: *Introduction to Crystallography,* W. A. Benjamin Advanced Book Program, New York (1969).

Schattschneider, Doris, and Walter Wallace: *M. C. Escher Kaleidocycles,* Ballantine, New York (1977).

Schuh, Fred: *The Master Book of Mathematical Recreations,* Dover, New York (1968).

Schuyt, M. and Joost Elffers: *Anamorphoses,* Abrams, New York (1975).

Seiden, Lloyd: *Buckminster Fuller's Universe,* Plenum, New York (1989).

Segal, Gerry: *Synergy Curriculum,* Board of Education, New York (1979).

Senechal, Marjorie and George Fleck, eds.: *Patterns of Symmetry,* Univ. of Massachusetts Press, Amherst (1977).

Senechal, Marjorie and George Fleck, eds.: *Structure of Matter and Patterns of Science,* Schenkman, Cambridge (1979).

Senechal, Marjorie and George Fleck, eds.: *Shaping Space: A Polyhedral Approach,* Birkhäuser Design Science Collection, A. L. Loeb, ed., Boston (1988).

Shahn, Ben: *The Shape of Content,* Vintage Books, New York (1957).

Sheppard, Roger, Richard Threadgill and John Holmes: *Paper Houses,* Schocken, New York (1974).

Shubnikov, A. V. and B. A. Koptsik: *Symmetry in Science and Art,* Plenum, New York (1972).

Smith, Cyril S.: *From Art to Science,* MIT Press, Cambridge (1980).

Smith, Cyril S.: *A Search for Structure,* MIT Press, Cambridge (1980).

Snyder, Robert: *Buckminster Fuller, An Autobiographical Monologue/Scenario,* St. Martin's Press, New York (1980).

Steinhaus, H.: *Mathematical Snapshots,* Oxford Univ. Press, New York (1969).

Stevens, Garry: *The Reasoning Architect: Mathematics and Science in Design,* McGraw Hill, New York (1990).

Stevens, Peter S.: *Patterns in Nature,* Little Brown, Boston (1973).

Stonerod, David: *Puzzles in Space,* Stokes, Palo Alto (1982).
Strache, Wolf: *Forms and Patterns in Nature,* Pantheon, New York (1956).

String Art Encyclopedia, Sterling, New York (1976).

Stuart, Duncan: *Polyhedral and Mosaic Transformations,* Univ. of North Carolina Press (1963).

Swirnoff, Lois: *Dimensional Color,* Birkhäuser Design Science Collection, A. L. Loeb, ed., Boston (1988).

Terpstra, P.: *Introduction to the Space Groups,* Univ. of Groningen Crystallographic Institute, Groningen, the Netherlands (1955).

Thom, Rene: *Stability and Morphogenesis,* Benjamin, Reading, Mass. (1975).

Thompson, d'Arcy Wentworth: *On Growth and Form,* Cambridge Univ. Press, London and New York (1942). Abridged edition by J. T. Bonner, Cambridge Univ. Press (1969).

Thompson, James B.: *SMART-Readings in Science and Art,* Red Clay Books, Charlotte, N.C. (1975).

Tompkins, Peter: *Secrets of the Great Pyramid,* Harper and Row, New York (1978).

Vasarely, Oeuvres Profondes Cinétiques, Ed. Griffon, Neuchatel (1973).

Verheyen, Hugo: *An Alternative Generation and Transformational Theory of Uniform Polyhedra,* Universitaire Instelling, Antwerp (1981).

Vilenkin, N. Y.: *Stories about Sets,* Academic Press, New York (1968).

Vinci, Leonardo da, Pamela Taylor, ed.: *The Notebooks of Leonardo da Vinci,* Plume Books, New York (1960).

Wachman, A., M. Burt and M. Kleinmann: *Infinite Polyhedra,* Technion, Haifa (1974).

Walter, Marion: *The Mirror Puzzle Book,* Tarquin, Norfolk, U.K. (1985).

Wechsler, Judith, ed.: *On Aesthetics in Science,* MIT Press (1978), reprint Birkhäuser Design Science Collection, A. L. Loeb, ed., Boston (1988).

Weisberger, Edward, ed.: *The Spiritual in Art: Abstract Painting 1890–1985,* Los Angeles County Museum of Art, Los Angeles (1986).

Wells, A. F.: *The Third Dimension in Chemistry,* Oxford Univ. Press, London and New York (1956).

Wells, A. F.: *Three-Dimensional Nets and Polyhedra,* Wiley, New York (1977).

Wells, A. F.: *Models in Structural Inorganic Chemistry,* Oxford Univ. Press, New York (1970).

Wenninger, Magnus: *Polyhedron Models,* Cambridge Univ. Press, London and New York (1971).

Wenninger, Magnus: *Spherical Models,* Cambridge Univ. Press, London and New York (1978).

Wenninger, Magnus: *Dual Models,* Cambridge Univ. Press, London and New York (1983).

Weyl, Hermann: *Symmetry,* Princeton Univ. Press, Princeton (1952).

Whyte, L. L., ed.: *Aspects of Form,* American Elsevier, New York (1968).

Whyte, L. L., A. G. Wilson and Donna Wilson, eds.: *Hierarchical Structures,* American Elsevier, New York (1969).

Wigner, Eugene P.: *Symmetries and Reflections,* Oxbow Press, Woodbridge, Conn. (1979).

Williams, Robert: *Natural Structure,* Eudaemon Press, Moorpark, Calif. (1972), reprint Dover, New York (1979).

Wilson, Forrest: *Architecture: A Book of Projects for Young Adults,* Van Nostrand Reinhold, New York (1968).

Winter, John: *String Sculpture,* Creative Publications, Palo Alto (1972).

Wittkower, Rudolf: *Idea and Image,* Thames and Hudson, London (1978).

Wolfe, Caleb Wroe: *Manual for Geometric Crystallography,* Edwards Bros., Ann Arbor (1953).

Wong, Wucius: *Principles of Three-dimensional Design,* Van Nostrand Reinhold, New York (1968 & 1977).

Wood, Donald G.: *Space Enclosure Systems,* Bulletins 203 and 205, Engineering Experiment Station, The Ohio State Univ., Columbus, Ohio (1973).

Yale, Paul: *Geometry and Symmetry,* Holden-Day, San Francisco (1968).

Young, Arthur M.: *The Reflexive Universe,* Delacorte, San Francisco (1976).

Zallgallaer, Viktor A.: *Convex Polyhedra with Regular Faces,* Plenum, New York (1969).

Zeier, Franz: *Paper Constructions,* Scribner, New York (1980).

ARTICLES:

Baracs, Janos, ed.: *La Revue Topologie Structurale,* published at irregular intervals, UQAM, C.P. 888, Succ. A, Montreal, Quebec, Canada.

Bolker, Ethan: *The Zonoid Problem,* Amer. Mathematical Monthly, *78,* 529–531 (1971).

Bolker, Ethan: *Transportational Polytopes,* J. Combinatorial Theory, *13,* 251–262 (1972).

Bolker, Ethan and Henry Crapo: *How to brace a One Story Building,* Environment and Planning, B4, 125–152 (1977).

Bolker, Ethan and Henry Crapo: *Bracing Rectangular Frameworks,* SIAM J. Appl. Math., *36,* 473–508 (1979).

Bolker, Ethan: *A Topological Proof of a Well Known Fact about Fibonacci Numbers,* The Fibonacci Quarterly, *15,* 245 (1977).

Bolker, Ethan: *Simplical Geometry and Transportation Polytopes,* Trans. Amer. Math. Soc., *217,* 121–142 (1976).

Bombieri, E., and J. Taylor: *Quasicrystals, Tilings, and Algebraic Number Theory: Some Preliminary Connections,* Contemp. Math., *64,* 241–264 (1987).

Bruijn, N. G. de: *Algebraic Theory of Penrose's Non-periodic Tilings of the Plane,* Kon. Nederl. Akad. Wetensch. Proc. Ser. A 84 (Indigationes Mathematicae 43), 38–66 (1981).

Bruijn, N. G. de: *Quasicrystals and their Fourier Transform,* Kon. Nederl. Akad. Wetensch. Proc. Ser. A 89 (Indigationes Mathematicae 48), 123–152 (1986).

Bruijn, N. G. de: *Modulated Quasicrystals,* Kon. Nederl. Akad. Wetensch. Proc. Ser. A 90 (Indigationes Mathematicae 49), 121–132 (1987).

Cahn, J. and J. Taylor: *An Introduction to Quasicrystals,* Contemp. Math., *64,* 265–286 (1987).

Coffin, Stewart T.: *Wooden Puzzles Easy to make, but Tough to Solve,* Fine Woodworking, November–December, 1984.

Coxeter, H. S. M.: *The Nine Regular Solids,* Proc. Canadian Math. Congress, Univ. of Toronto Press (1946).

Coxeter, H. S. M.: *Close Packing and Froth,* Illinois J. of Mathematics, *2,* 746–758 (1958).

Cundy, H. Martyn: *Deltahedra,* The Mathematical Gazette, *36,* 263–266 (1952).

Delone, B., N. Dolbilin, M. Shtogrin and R. Galiulin: *A Local Criterion for the Regularity of a System of Points,* Soviet Math. Dokl., *17,* 319–322 (1976).

Diaconis, Persi and Josephine B. Keller: *Fair Dice,* American Mathematical Monthly, *96* (April 1989).

Ede, John D.: *Rhombic Triacontahedra,* The Mathematical Gazette, *36,* no. 340, 98.

Erickson, R. O.: *Phyllotaxis and Shoot Form,* Botanical Soc. of Amer., Minneapolis (1972).

Erickson, R. O.: *Are Pith Cells 14-sided?,* Bot. Soc. Amer., Fort Collins, Colo. (1984).

Erickson, R. O.: *Tubular Packing of Spheres in Biological Fine Structure,* Science, *181,* 705–716 (1973).

Erickson, R. O.: *The Geometry of Phyllotaxis,* in J. E. Dale and F. L. Milthorp, eds.: *The Growth and Functioning of Leaves,* Cambridge Univ. Press, New York (1982).

Erickson, R. O., and Alburt Rosenberg: *Phyllotaxis of the Sunflower Head,* Bot. Soc. Amer., Fort Collins, Colo. (1984).

Fejes Toth, L.: *What the Bees know and what they do not know,* Bulletin of the American Mathematical Society, *20,* 468–481 (1964).

Frank, F. C., and J. S. Kasper: *Complex Alloy Structures Regarded as Sphere Packings,* Acta Crystallographica, *11,* 184–190 (1957), and *12,* 483–499 (1958).

Gardner, Martin: *Mathematical Games: Extraordinary nonperiodic tiling that enriches the theory of tiles,* Scientific American, January 1977.

Godrèche, C., and J. M. Luck: *Quasiperiodicity and Randomness in Tilings of the Plane,* J. Stat. Phys., *55,* 1–28 (1989).

Goldberg, Michael: *Several New Space Filling Polyhedra,* Geometriae Dedicata, *5,* 517 (1976).

Goldberg, Michael: *On the Space Filling Hexahedra,* Geometriae Dedicata, *6,* 99 (1977).

Goodenough, J. B., and A. L. Loeb: *A Theory of Ionic Ordering, Tetragonal Phase Formation, Magnetic Exchange and Lamellar Precipitation due to Covalent Forces in Spinels,* Phys. Rev., *98,* 391–408 (1955).

Gridgeman, N. T.: *Platonic Sphericity,* Recreational Mathematics, *6,* no. 3, 206 (1973).

Grünbaum, Branko: *Regular Polyhedra—Old and New,* Israel J. of Math., *1,* 191–197 (1963).

Grünbaum, Branko, with N. W. Johnson: *The Faces of a Regular-faced Polyhedron,* J. London Math. Soc., *20,* 577–586 (1965).

Grünbaum, Branko: *Polytopes, Graphs, and Complexes,* Bull. Amer. Math Soc., *76,* 1131–1201 (1970).

Grünbaum, Branko, and G. C. Shepard: *Satins and Twills: an Introduction to the Geometry of Fabrics,* Mathematics Magazine, *53,* 131–161 and 313 (1980).

Grünbaum, Branko, and G. C. Shepard: *A Hierarchy of Classification Methods for Patterns,* Zeitschrift f. Kristallographie, *154,* 163–187 (1981).

Grünbaum, Branko, and G. C. Shepard: *Spherical Tilings with transitivity properties,* in *The Geometric Vein: The Coxeter*

Festschrift, C. Davis, B. Grünbaum and F. Shenk, eds., Springer Verlag, New York (1982).

Grünbaum, Branko, G. C. Shepard and L. Danzer: *Does Every Type of Polyhedron Tile Three-space?,* Structural Topology, *8,* 3–14 (1983).

Harris, W. F., and R. O. Erickson: *Model Biological Tubules: the Geometry and Construction of Tubular Arrays of Spheres,* Lecture Notes in Biomath., *33,* 175–191 (Springer, New York).

Harris, W. F., and R. O. Erickson: *Tubular Arrays of Spheres; Geometry, Continuous and Discontinuous Contraction, and the Role of Moving Dislocations,* J. Theor. Biology, *83,* 215–246 (1980).

Haughton, Eric C., and A. L. Loeb: *Symmetry: A Case History of a Program,* J. Research in Science Teaching, *2,* 132ff (1964).

Henley, Christopher L.: *Quasicrystal Order, its Origins and its Consequences: A Survey of Current Models.* Comments on Condensed Matter Physics, 58–117 (1987).

Hoffer, William: *A Magic Ratio Recurs Throughout Art and Nature,* Orion Nature Quarterly, *4,* no. 1, 28–38 (1985).

Hope, C.: *The Nets of the Regular Star-Faced and Star-Pointed Polyhedra,* The Mathematical Gazette, *35,* 8–11 (1951).

Horgan, John: *Quasicrystals, Rules of the Game,* Science, *247,* March 2 (1990).

James, Albert: *Polyhedral Architecture,* Architectural Association Quarterly, *4,* (July–September, 1972).

James, Albert: *The Geometry of my Polyhedral Sculpture,* Leonardo, *10,* 183–187 (1977).

James, Albert: *The Cube and the Dodecahedron in my Polyhedron Architecture,* Leonardo, *13,* 272–275 (1980).

Katz, A., and M. Duneau: *Quasiperiodic Patterns and Icosahedral Symmetry,* J. de Physique, *47,* 181–196 (1986).

Katz, A.: *Theory of Matching Rules for the 3-dimensional Penrose Tilings,* Commun. Math. Phys., *118,* 263–288 (1988).

Klug, A.: *Architectural Design of Spherical Viruses,* Nature, *303,* 378–379 (1983).

Krull, Wolfgang, transl. Betty S. Waterhouse and William C. Waterhouse: *The Aesthetic Viewpoint in Mathematics,* The Math. Intelligencer, *9,* no. 1, Springer, New York (1987).

La Brecque, M.: *Opening the Door to Forbidden Symmetries,* Mosaic (Nat'l Science Foundation), *18,* 2–23 (1987–'88).

Levitov, L., and J. Rhyner: *Crystallography of Quasicrystals: Applications to Icosahedral Symmetry,* J. Phys. France, *49,* 1835–1849 (1988).

Loeb, A. L.: *A Binary Algebra Describing Crystal Structures with Closely Packed Anions,* Acta Cryst., *11,* 469–476 (1958).

Loeb, A. L.: *A Modular Algebra for the Description of Crystal Structures,* Acta Cryst., *15,* 219–226 (1962).

Loeb, A. L., and G. W. Pearsall: *Moduledra Crystal Models,* Amer. J. Phys., *31,* 190–196 (1963).

Loeb, A. L.: *The Subdivision of the Hexagonal Net and the Systematic Generation of Crystal Structures,* Acta Cryst., *17,* 179–182 (1964).

Loeb, A. L.: *Remarks on some Elementary Volume Relations between Familiar Solids,* Mathematics Teacher, *50,* 417 (1965).

Loeb, A. L., and E. C. Haughton: *The Programmed Use of Physical Models,* J. Programmed Instruction, *3,* 9–18 (1965).

Loeb, A. L.: *The Architecture of Crystals,* in Gyorgy Kepes, ed. *Module, Proportion, Symmetry, Rhythm,* Vision and Value Series, Braziller, New York (1966).

Loeb, A. L.: *Structure and Patterns in Science and Art,* Leonardo, *4,* 339–345 (1971) reprinted in *Visual Art, Mathematics and Computers,* 5–11, ed. Frank J. Malina, Pergamon Press (1979).

Loeb, A. L.: *Comments on David Drabkin's Book: Fundamental Structure: Nature's Architecture,* Leonardo, *10,* 313–314 (1977).

Loeb, A. L.: *Algorithms, Structure and Models,* in *Hypergraphics,* AAAS Selected Symposium Series, 49–68 (1978).

Loeb, A. L.: *A Studio for Spatial Order,* Proc. Internat'l Conf. on Descriptive Geometry and Engineering Graphics, Engineering Graphics Division, Amer. Soc. Engineering Education , 13–20 (1979).

Loeb, A. L.: *Escher, A Review of the book edited by J. L. Locher,* London Times Literary Supplement, September 17, 1982.

Loeb, A. L.: *Natural Structure in the Man-made Environment,* The Environmentalist, *2,* 43–49 (1982).

Loeb, A. L.: *On my Meetings and Correspondence between 1960 and 1971 with Graphic Artist M. C. Escher,* Leonardo, *15,* 23–27 (1982).

Loeb, A. L.: *Synergy, Sigmoids and the Seventh-Year Trifurcation,* The Environmentalist, *3,* 181–186 (1983). Reprinted in Chrestologia, XIV no. 2, 4–8 (1989).

Loeb, A. L.: *Vector Equilibrium Synergy,* J. Space Structures, *1,* 99–103 (1985).

Loeb, A. L.: *Symmetry and Modularity,* J. Computers & Mathematics with Applications, *128,* Nos. 1/2, 63–75 (1986). Reprinted in *Symmetry, Unifying Human Understanding,* ed. Istvan Hargittai, Pergamon Press, New York, 63–75 (1986) in Internat'l Series in Modern Applied Mathematics & Computer Science, Ervin Y. Rodin, Series Editor.

Loeb, A. L.: *Symmetry in Court and Country Dance,* J. Comp. & Math. with Appl., 629–639 (1986). Reprinted as above, 629–639 (1986).

Loeb, A. L.: *Polyhedra in the Work of M. C. Escher,* in *M. C. Escher: Art and Science,* ed. H. S. M. Coxeter et alia, North Holland (195–202) (1986).

Loeb, A. L.: *Color and Symmetry: Ordering the Environment with Pattern,* Color and Light, *17,* 16–18 (1987).

Loeb, A. L.: *The magic of the pentangle: Dynamic Symmetry from Merlin to Penrose,* J. Computers & Mathematics with Applications, *17,* 33–48 (1989).

Loeb, A. L., J. C. Gray and P. R. Mallinson: *On the Icosahedron, the Pentagonal Dodecahedron, and the Rhombic Triacontahedron,* Symmetry, *1,* 29–36 (1990).

Loeb, A. L.: *Forces Shaping the Spaces we Inhabit,* J. Space Structures, *To be published* (1991).

Loeb, A. L., and William Varney: *A Stabilized Cuboctahedron Frame,* J. Space Structures, *To be published* (1991).

Loeb, A. L.: *On Behaviorism, Causality and Cybernetics,* Leonardo, *To be published* (1991).

Loeb, A. L., and William Varney: *Does the Golden Spiral Exist, and if Not, Where is its Center?* in *Spiral Symmetry,* Istvan Hargittai, ed., World Science Books, *To be published* (1991).

Mackay, A.: *Crystallography and the Penrose Pattern,* Physics 114A, 609–613 (1982).

Masunaga, David: *Making Mathematics Appeal to the Mind and the Eye,* Harvard Grad. Sch. of Education Alumni Bull., Fall–Winter 1987, 18–21.

McGowan, William E.: *A Recursive Approach to Construction of the Deltahedra,* Mathematics Teacher, March 1978, 204–210.

Miekisz, J., and C. Radin: *Are Solids Really Crystalline?,* Phys. Rev. B39, 1950–1952 (1989).

Monson, Barry: *The Densities of Certain Regular Star Polytopes,* C. R. Math. Rep. Academy of Science in Canada, *2,* 73–78 (1980).

Monson, Barry: *The Schlaflian of a Crystallographic Coxeter Group,* C. R. Math. Rep. Academy of Science in Canada, *4,* 145–147 (1982).

Monson, Barry: *Uniform Polyhedra from Diophantine Equations,* in *Shaping Space,* M. Senechal and G. Fleck, eds., 219–220, Birkhäuser Design Science Collection, A. L. Loeb, ed., Boston (1988).

Morris, I. L. and A. L. Loeb: *A Binary Algebra Describing Crystal Structures with Closely Packed Anions, Part II: A Common System of Reference for Cubic and Hexagonal Structures,* Acta Cryst., *13,* 434–443 (1960).

Nelson, David R.: *Quasicrystals,* Scientific American, *255,* 42 (1986).

Pawley, G. S.: *The 227 Triacontahedra, 4,* no. 221 (1975).

Penrose, Roger: *The role of aesthetics in pure and applied mathematics research,* J. Inst. Mathematics and its Applications, *10,* 266–271 (1974).

Rivier, N.: *On the Structure of Random Tissues or Froths, and their Evolutions,* Philosophical Magazine, series 8, *47,* L45–L49 (1983).

Schattschneider, Doris: *Tiling the Plane with Congruent Pentagons,* Math Magazine, *51,* 29–44 (1980).

Schulte, Egon: *Tiling Three-space by Combinatorially Equivalent Convex Polytopes,* Proc. London Math. Soc. (3), *49,* 128–140 (1984).

Senechal, Marjorie: *Which Tetrahedra Fill Space?,* Mathematics Magazine, *54,* 227–243 (1981).

Senechal, Marjorie, and Jean Taylor: *Quasicrystals: The View from Les Houches,* The Mathematics Intelligencer, *12,* no. 2, 54–64 (1990).

Shechtman, D., I. Blech, D. Gratias and J. Cahn: *Metallic Phase with Long-range Orientational Order and no Translational Symmetry,* Phys. Rev. Letters, *53,* 1951–1954 (1984).

Smith, Anthony: *Stellations of the Triakis Tetrahedron,* The Math. Gazette, *49,* no. 368, 135.

Smith, Cyril S.: *Grain Shapes and Other Metallurgical Applications of Topology,* Metal Interfaces, Amer. Soc. for Metals; University Microfilms, Ann Arbor, Mich., OP 13754, 65–113.

Somerville, D. M. Y.: *Space-filling Tetrahedra in Euclidean Space,* Proc. Edinburgh Math. Soc., *41,* 49 (1923).

Stogrin, M. I.: *Regular Dirichlet-Voronoi Partitions for the Second Triclinic Group,* Proc. Steklov Inst. of Math., *123* (1975), Amer. Math. Soc., Providence, R.I. (1975).

Tarnai, Tibor: *Spherical Grids of Triangular Network,* Acta Technica Hungarian Academy of Sciences, Budapest (1974).

Von Baeyer, Hans C.: *Impossible Crystals,* Discover, February 1990.

Wachman, Avram, and Alexander Levitzki: *Mechanism for the Assembly of Spherical Viruses: a Geometrical Approach,* J. Theoretical Biology, *34,* 277–287 (1972).

Wenninger, Magnus J.: *Some Interesting Octahedral Compounds,* Math. Gazette, *52,* 10 (1968).

Wenninger, Magnus J.: *A New Look for the Old Platonic Solids,* Summation, J. Assoc. Teachers of Math., New York (Winter 1971).

Wenninger, Magnus J.: *Geodesic Domes by Euclidean Construction,* Math. Teacher (October 1978).

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