Tilings of the hyperbolic space and their visualization

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Corvallis, Oregon, USA
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Abstract

Visual representation of tiling of 3D hyperbolic space attracted very little attention compare to tilings of hyperbolic plane, which were popularized by M.C.Escher circle limit woodcuts. Although there is a lot of activity on theoretical side of the problem starting from work of H.Poincare on Kleinian groups and continuing with breakthrough of W.Thurston in the development of low dimensional topology and G.Perelman's proof of Poincare conjecture.

The book "Indra's Pearl" have popularized visualization of 2D limit set of Kleinian groups, which is located at the infinity of hyperbolic space. In this talk we present our attempts to build and visualize actual 3D tilings. We study tilings with symmetry group generated by reflections in the faces of Coxeter polyhedron, which also is the fundamental polyhedron of the group.

Online address of the talk:
bulatov.org/math/1101/

Outline

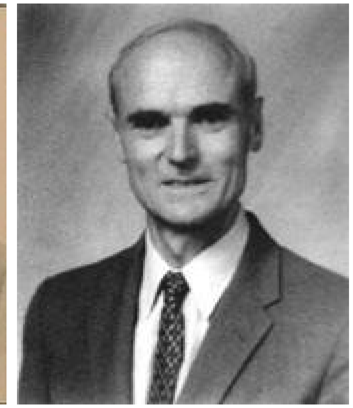
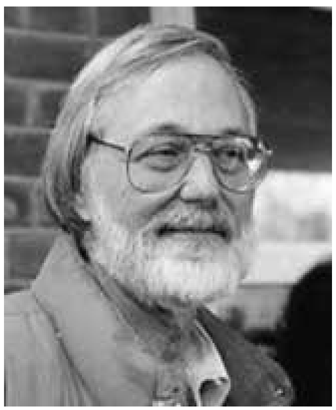
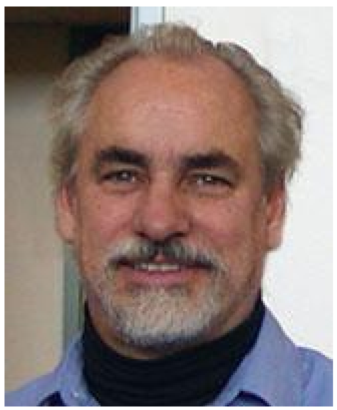
- Introduction. Who, what and when
- 2D tiling with Coxeter polygons
- 3D tiling with Coxeter polyhedra
- Hyperbolic polyhedra existence and construction
- Building isometry group from generators
- Visualization of the group and it's subgroups
- Tiling zoo
- Implementation in metal

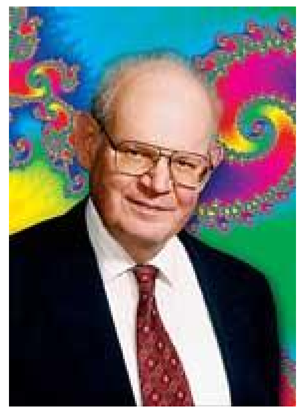
Introduction

These are some of the people involved in the development of the hyperbolic geometry and classification of symmetry groups of hyperbolic plane and space:
N.Lobachevski, Janos Bolyai, F.Klein, H.Poincare, D.Coxeter, J.Milnor, J.Hubbard, W.Thurston, B.Mandelbrot, G.Perelman

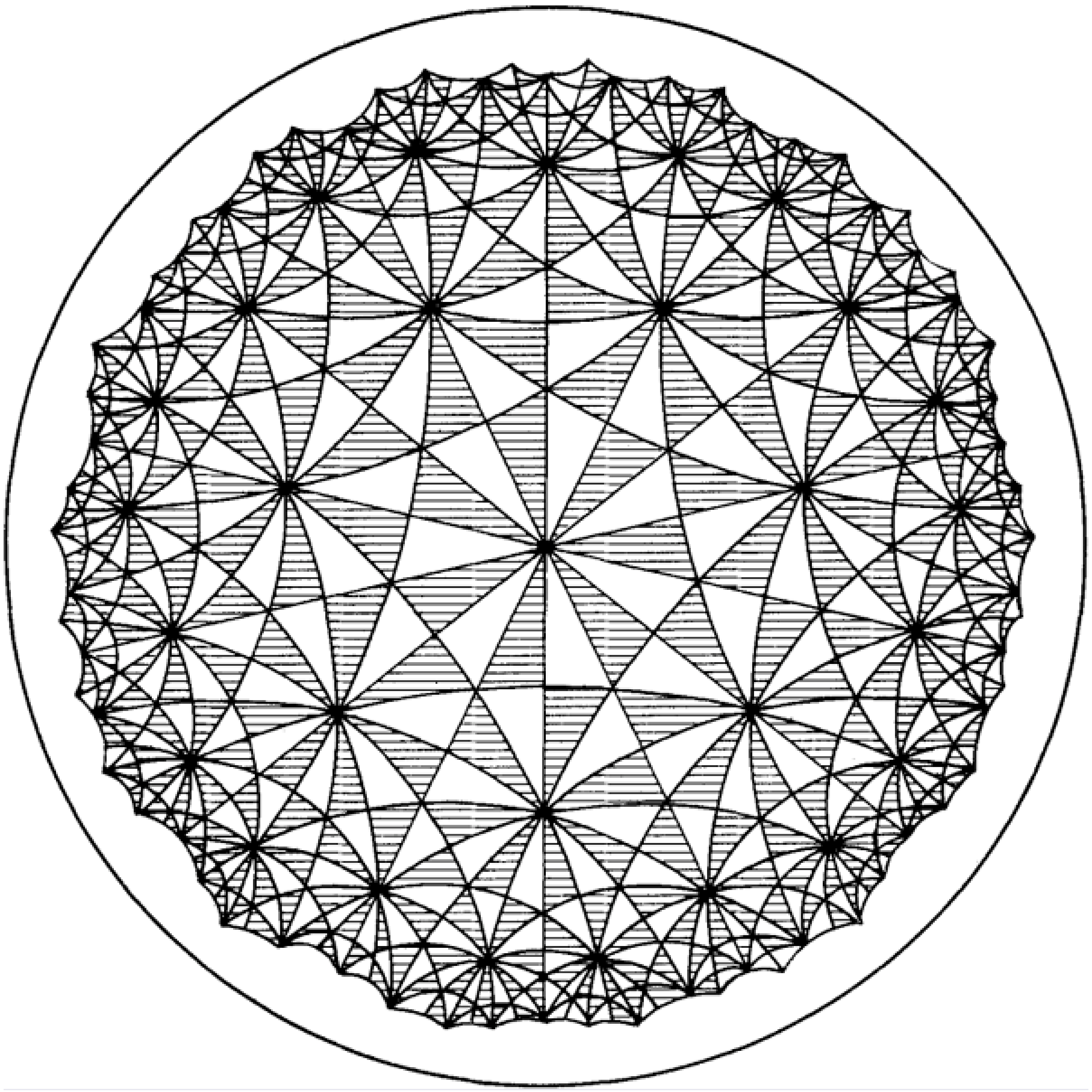




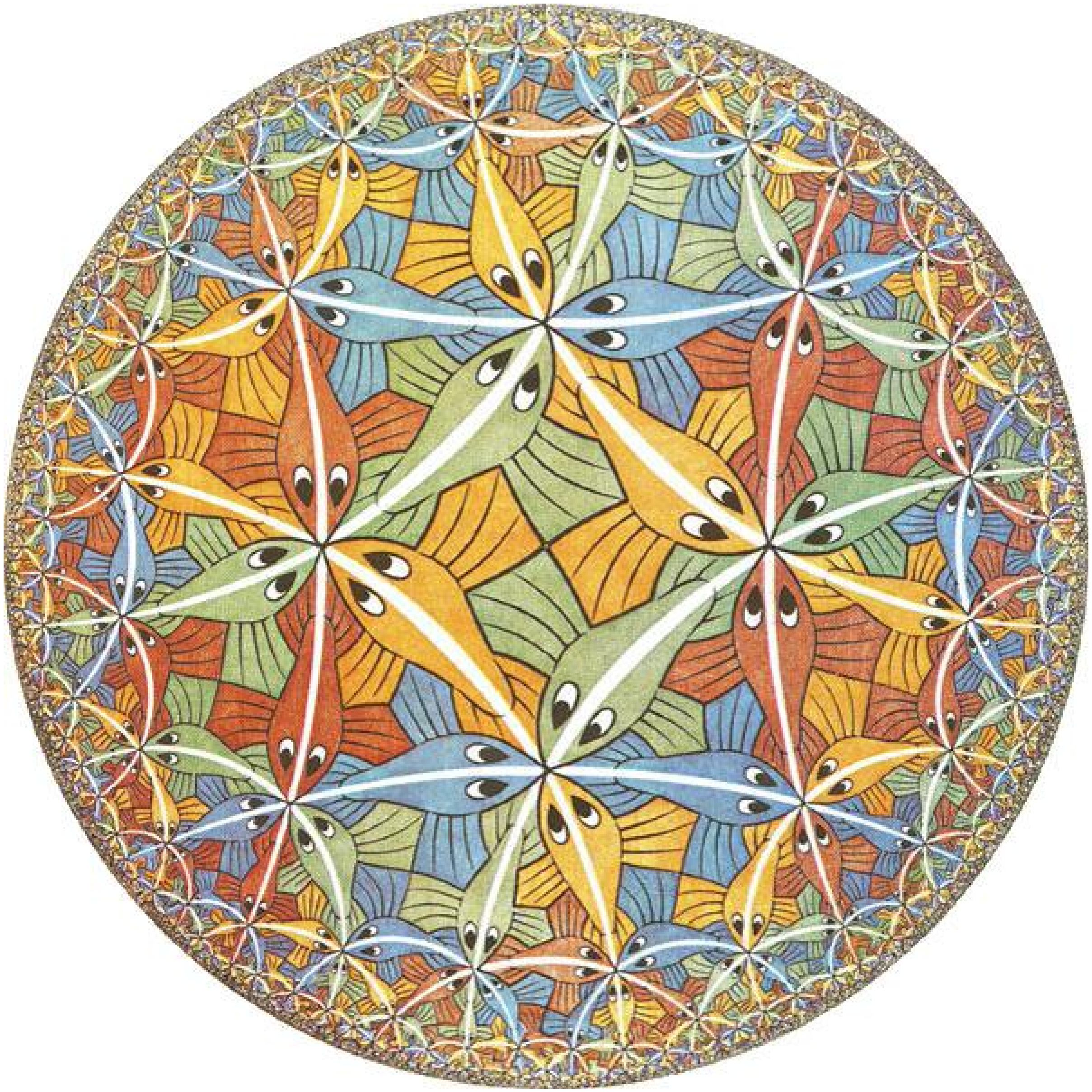



Visualization of 2D hyperbolic tiling



The first high quality drawing of 2D hyperbolic tiling apperas 1890 in book by Felix Klein and Robert Fricke.
This is drawing of *237 tessellation from F.Klein and R.Fricke "Vorlesungen über die Theorie der Elliptischen Modulfunctionen," Vol. 1, Leipzig(1890)

Visualization of 2D hyperbolic tilings



M.C.Escher Circle Limit III (1958)
He was inspired by drawing of hyperbolic tiling in paper of H.S.M.Coxeter.

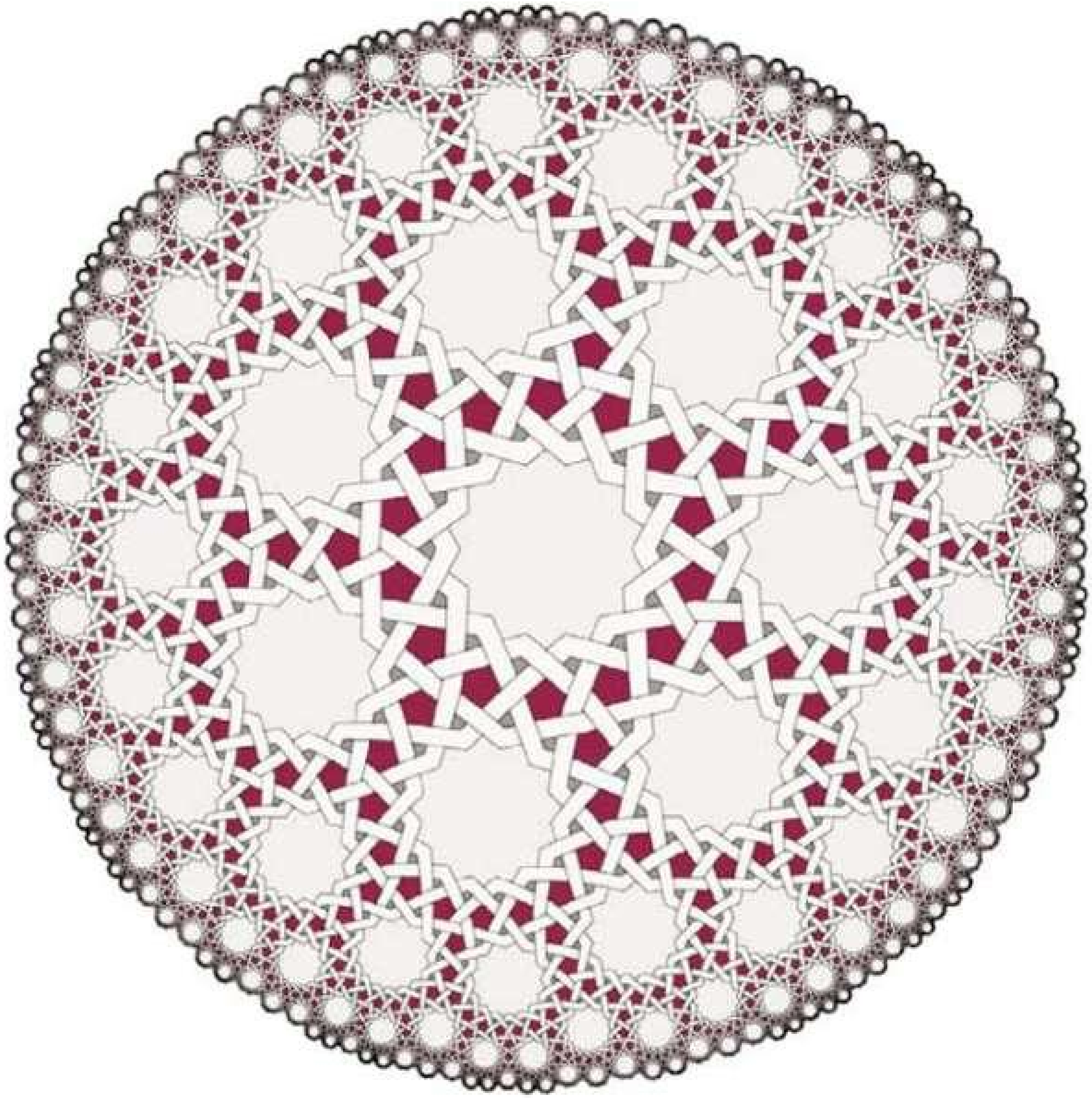
Visualization of 2D hyperbolic tilings

Many hyperbolic patterns by D.Dunham

Visualization of 2D hyperbolic tilings

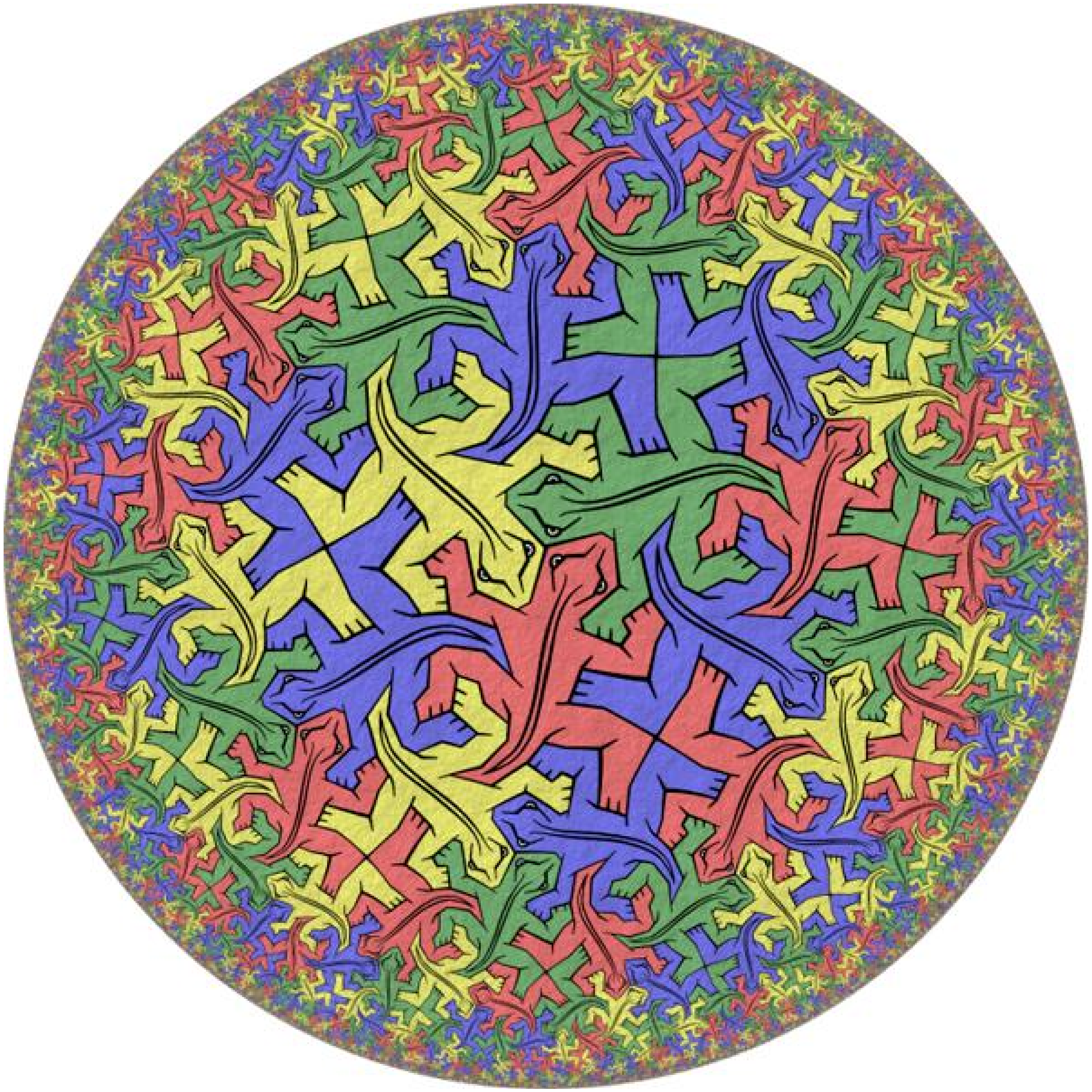
C.Goodman-Strauss's graphics work.

Visualization of 2D hyperbolic tilings



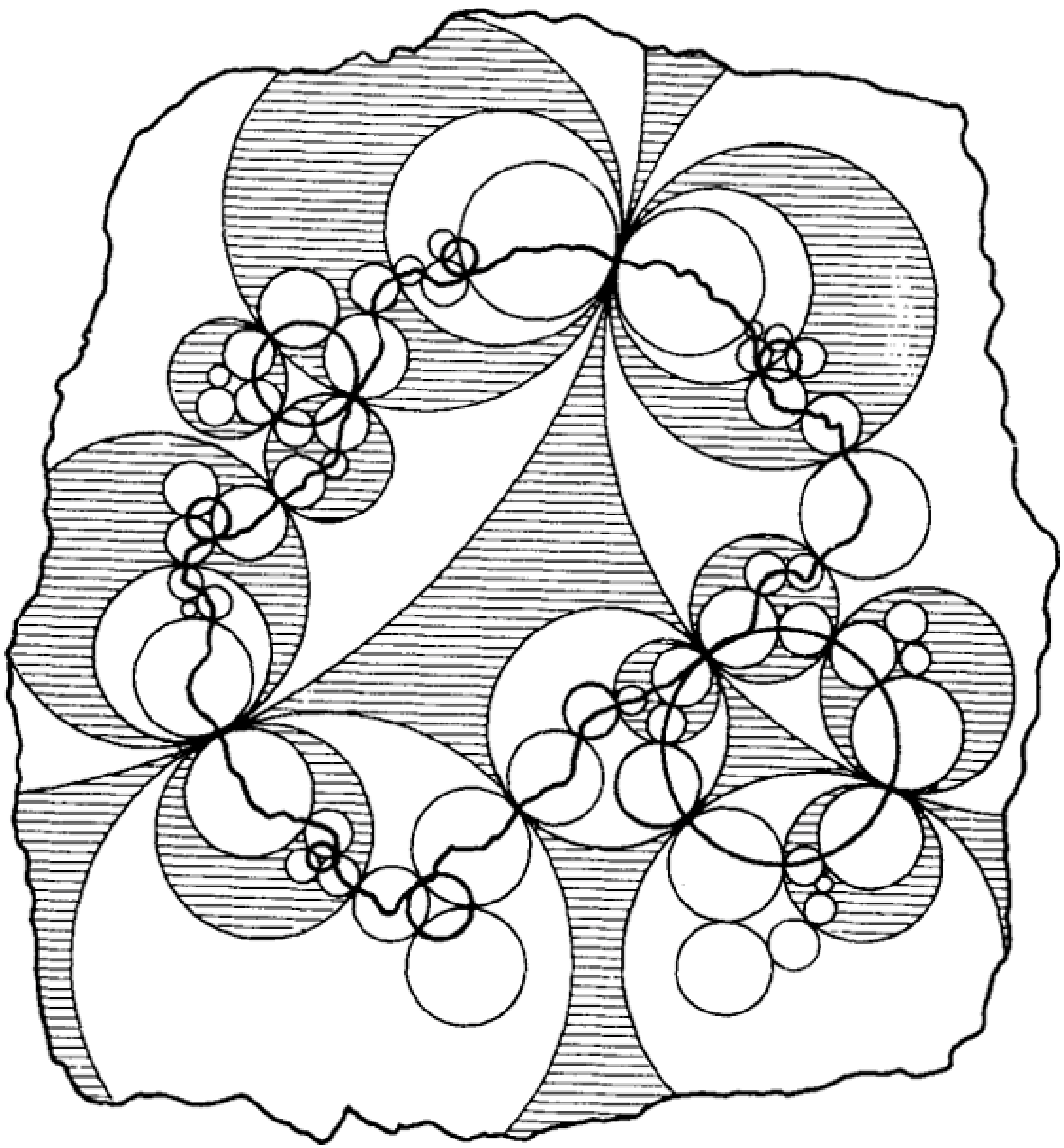
C.Kaplan's Islamic pattern.

Visualization of 2D hyperbolic tilings



Martin von Gagern "Hyperbolization of Euclidean Ornaments"

Limit set of Kleinian groups

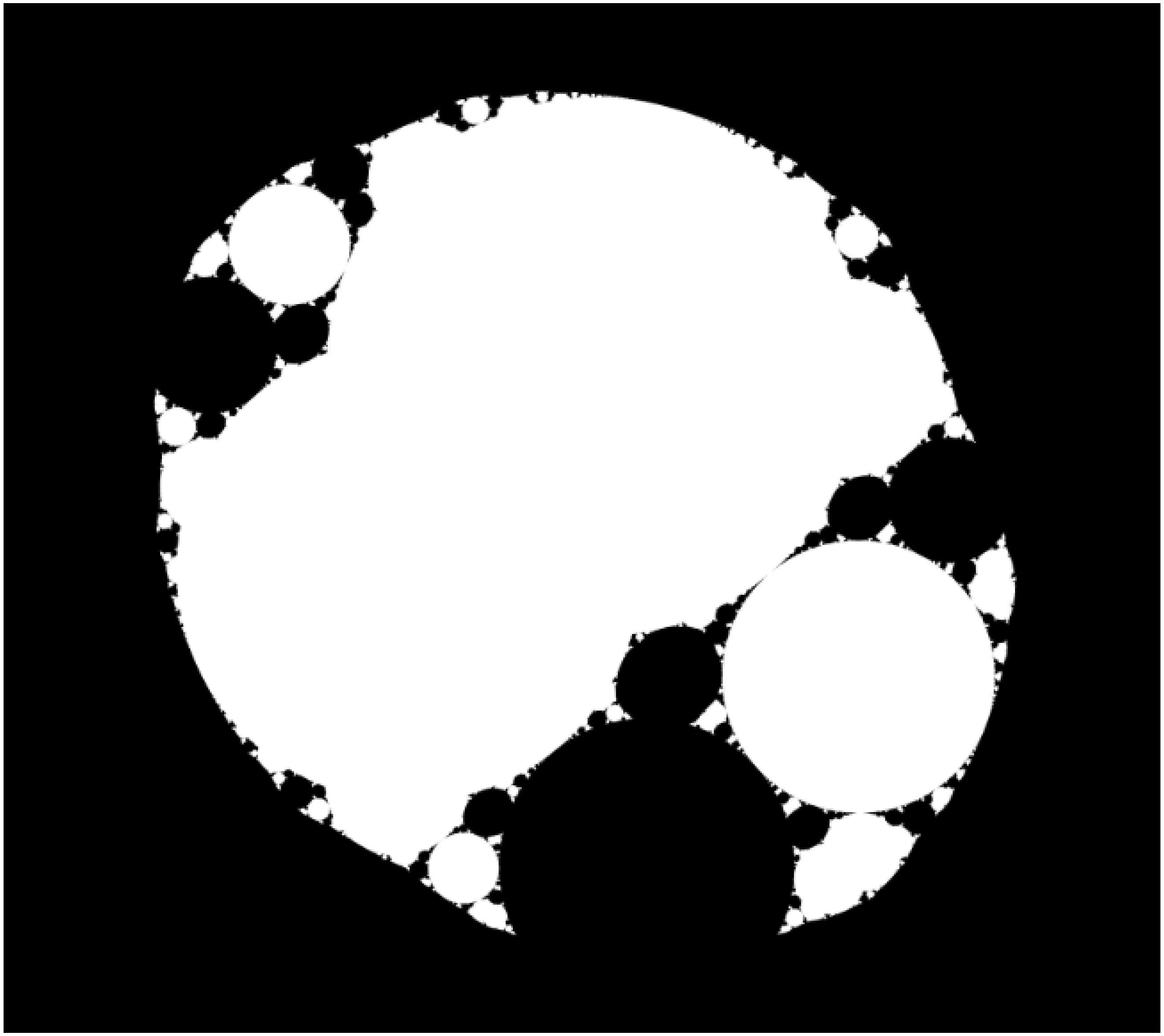


3D tilings were visualized by rendering it's 2D limit set on the infinity of the hyperbolic space.

One of the first know drawings of limit set from F.Klein and R.Fricke "Vorlesungen über die Theorie der Automorphen Functionen" Leipzig(1897)

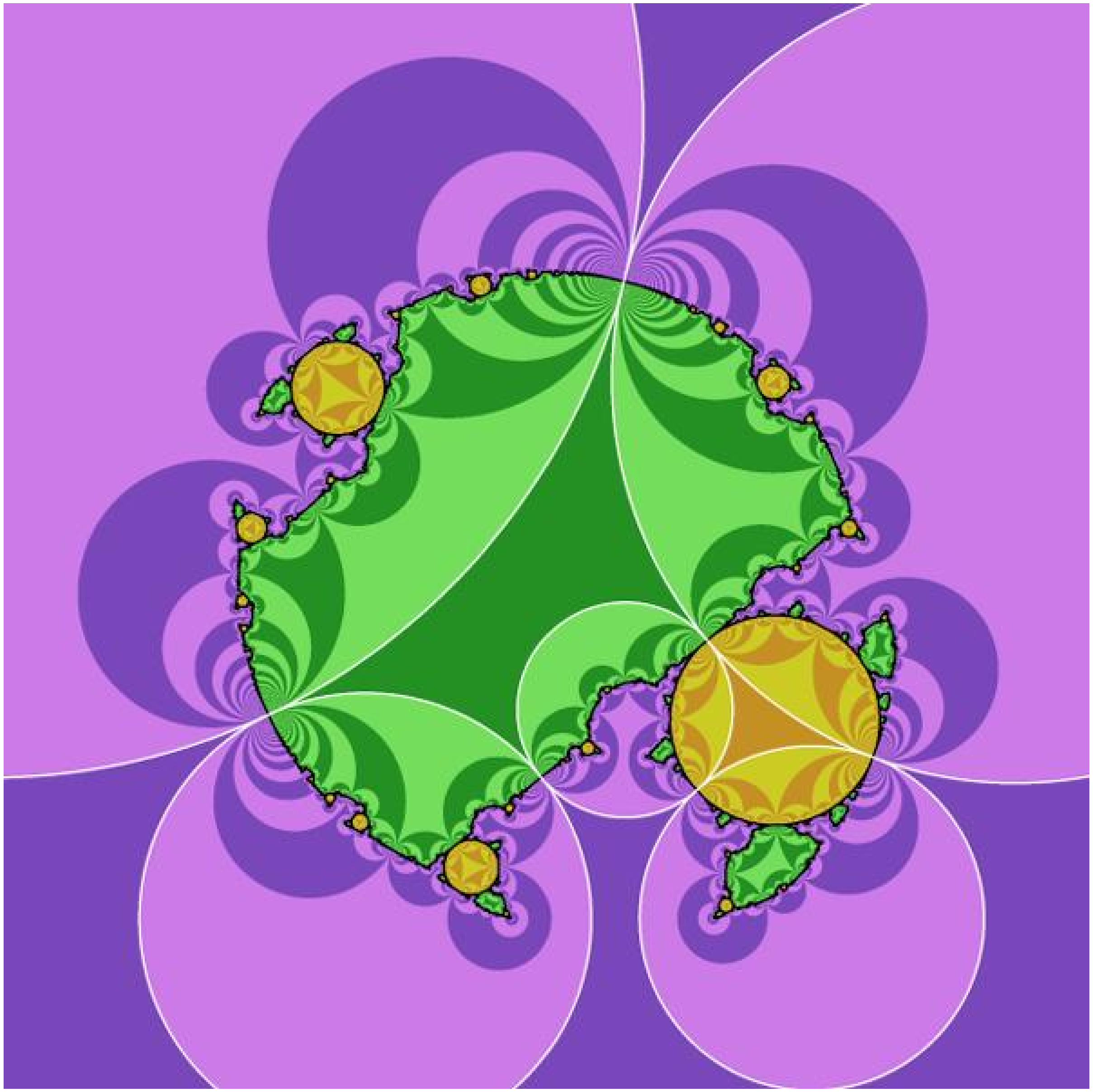
This hand drafted image was the best available to mathematicians until 1970s, when B.Mandelbrot started to make computer renderings of Kleinian groups

Limit set of Kleinian groups



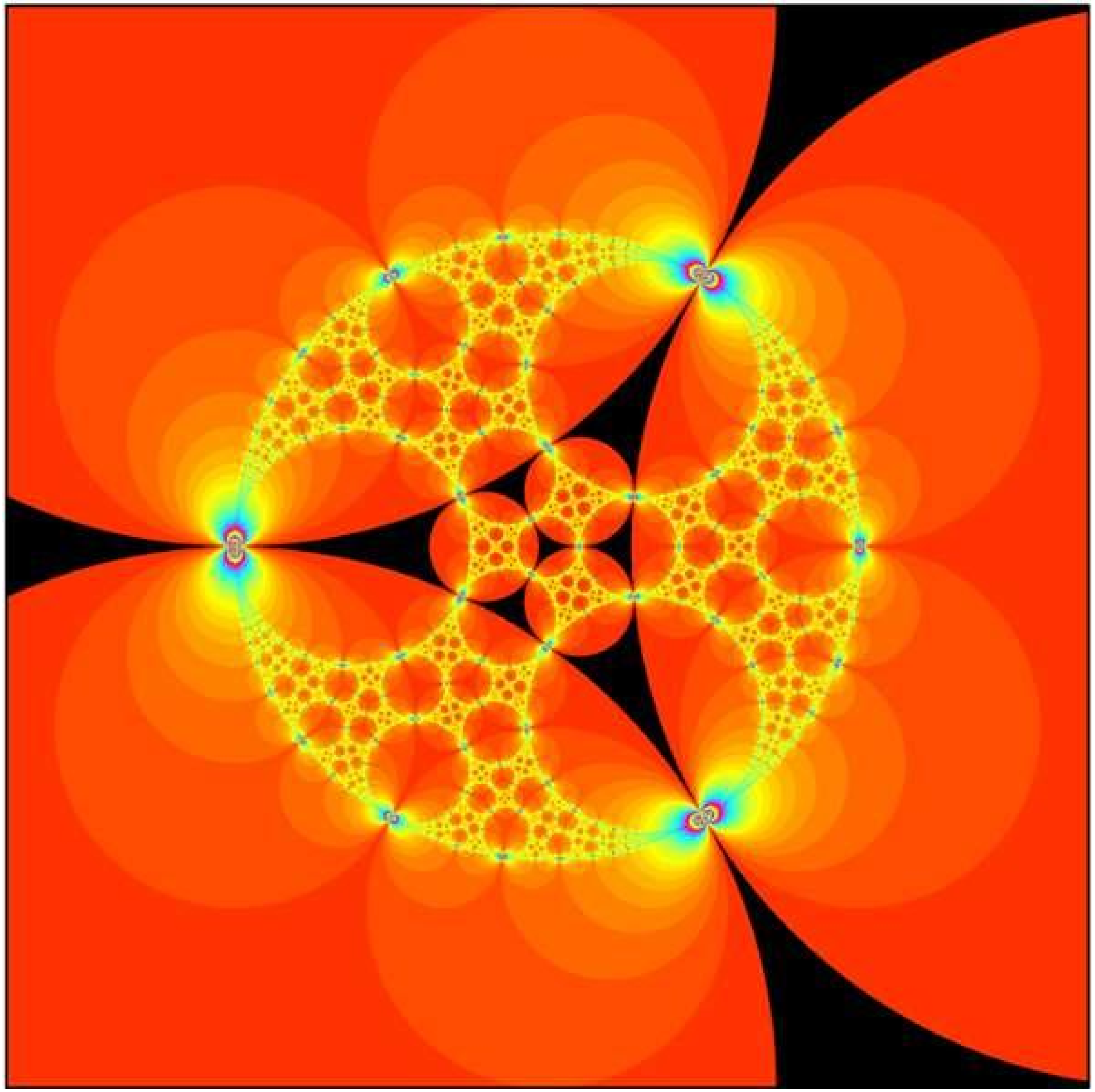
B.Mandelbrot's computer rendering of Kleinian group (from Un ensemble-limite par Michele Audin et Arnaud Cheritat (2009))

Limit set of Kleinian groups



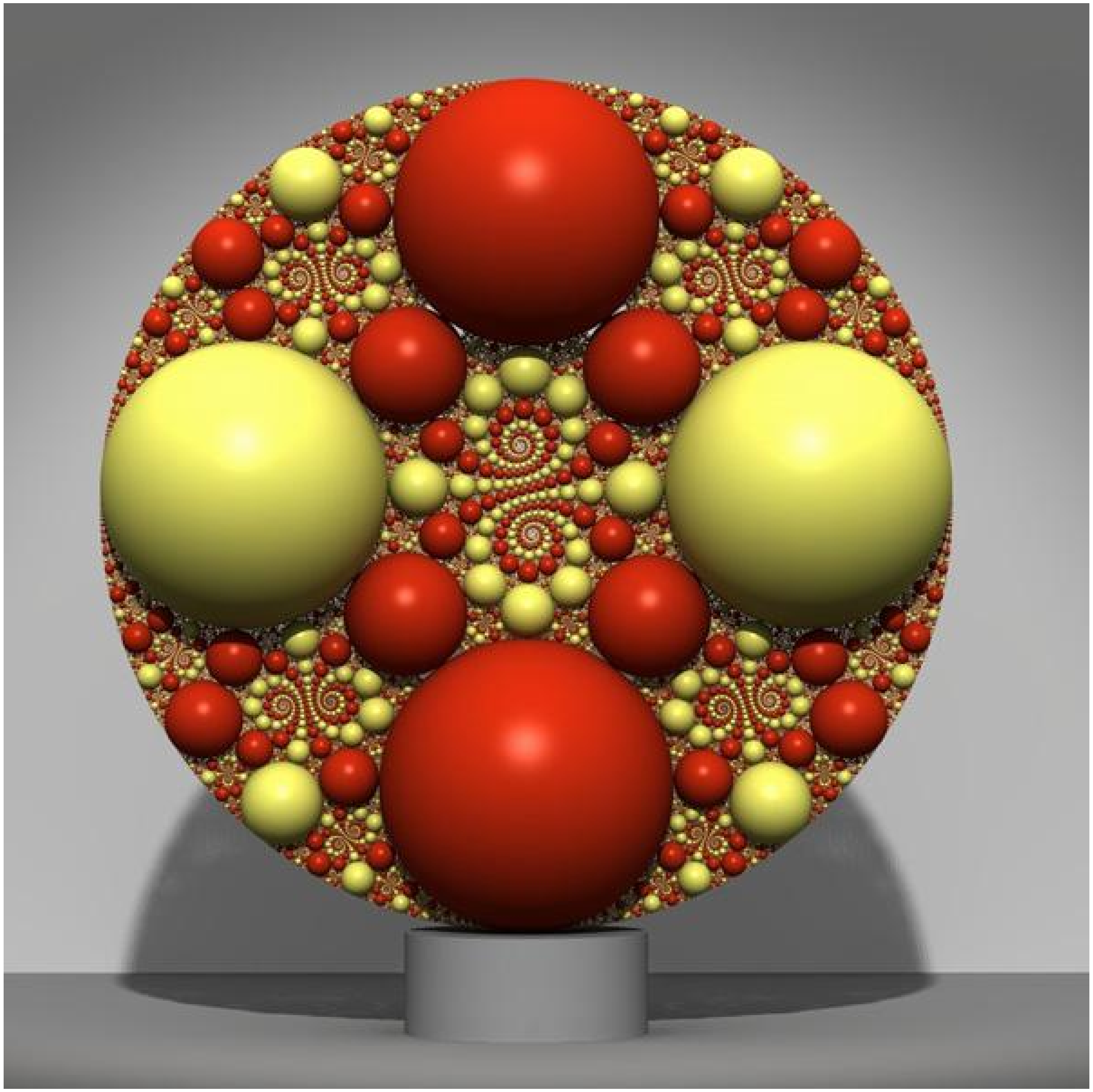
Interactive applet of the kleinian group visualization by Arnaud Cheritat. From Un ensemble-limite.

Limit set of Kleinian groups



"Indra Pearl" by D.Mumford, C.Series and D.Wright (2004) popularized visualization of limit sets of Kleinian groups

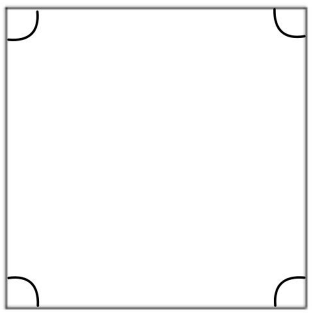
Limit set of Kleinian groups



Kleinian group images by Jos Leys.

His very artistic images rendered as 3D images represent essentially 2-dimensional limit set of Kleinian group.

Tiling with Coxeter Polygons



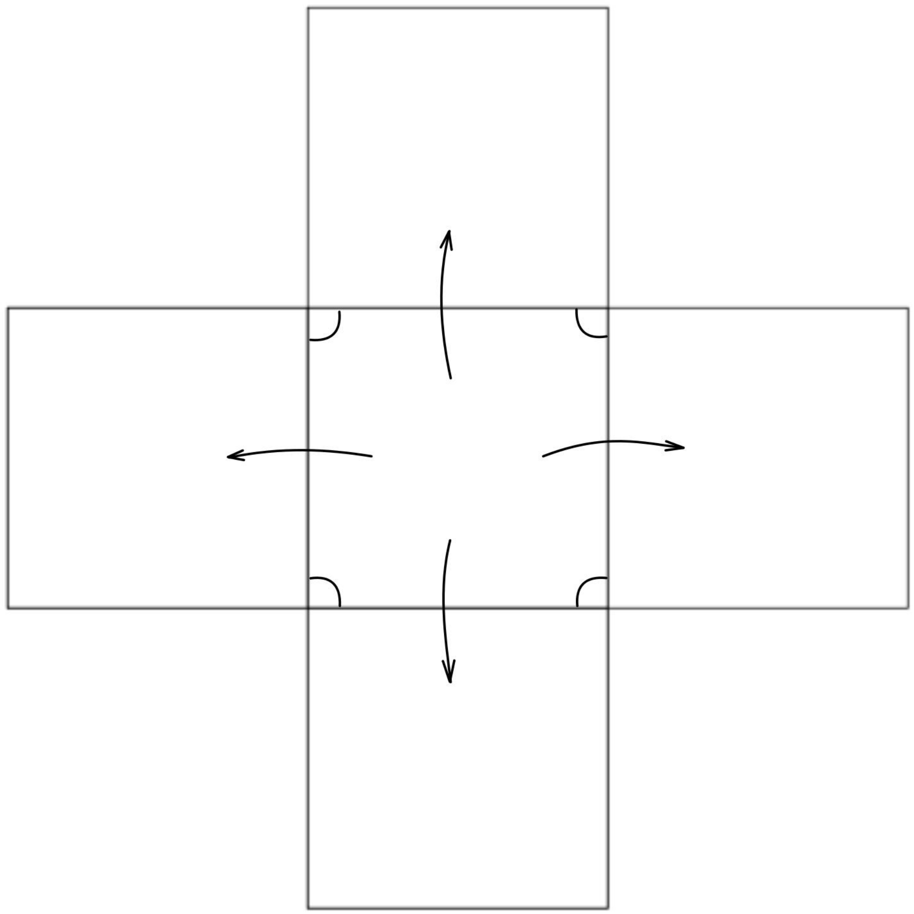
Coxeter polygon:

All angles are submultiples of π

$\alpha_i = \dfrac{\pi}{m_i}$.

Here all four angles of euclidean polygon are equal $\dfrac{\pi}{2}$

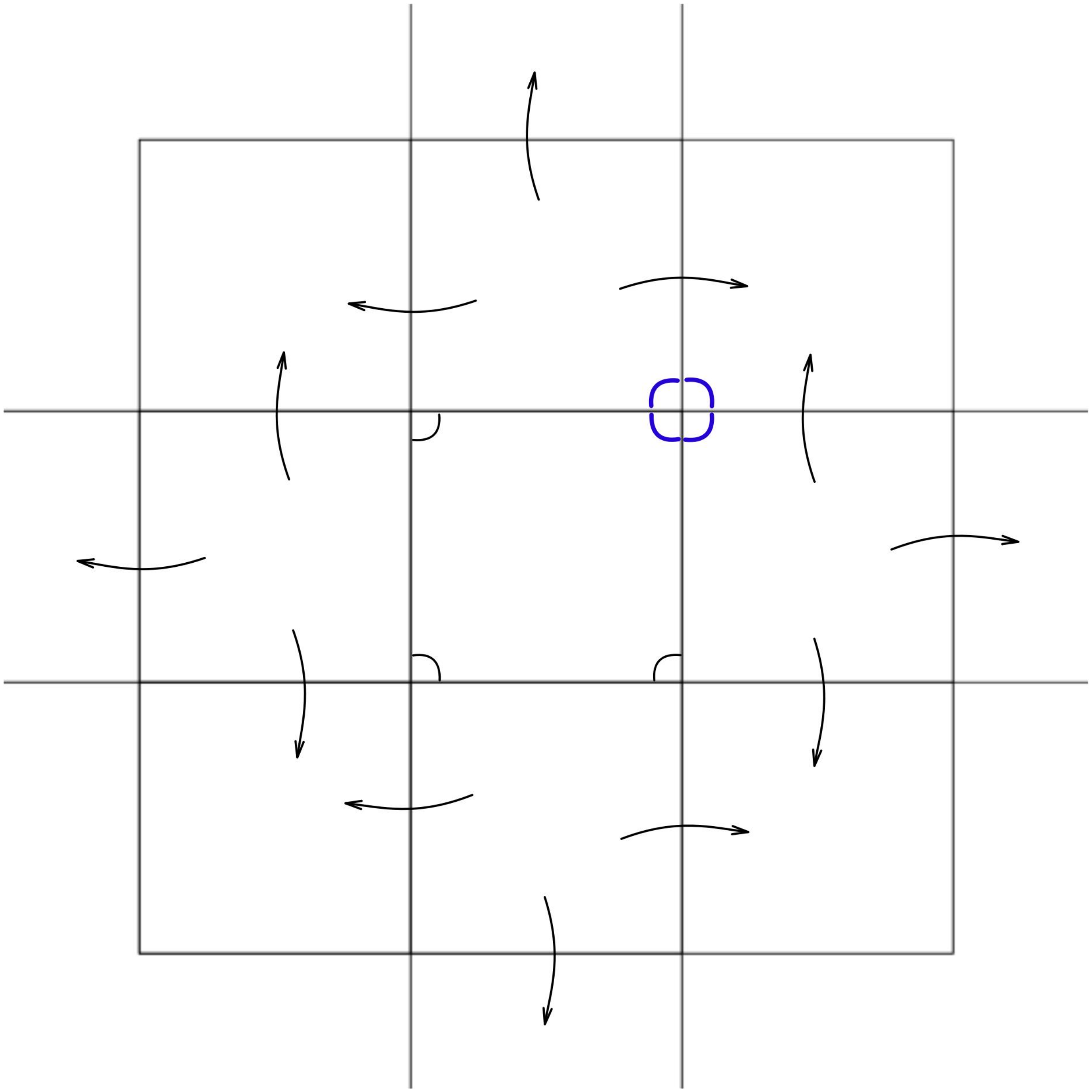
Tiling with Coxeter Polygons



The tiling is generated by reflections in the sides of the Coxeter polygon.

Here we have original tile and second generation of tiles.

Tiling with Coxeter Polygons

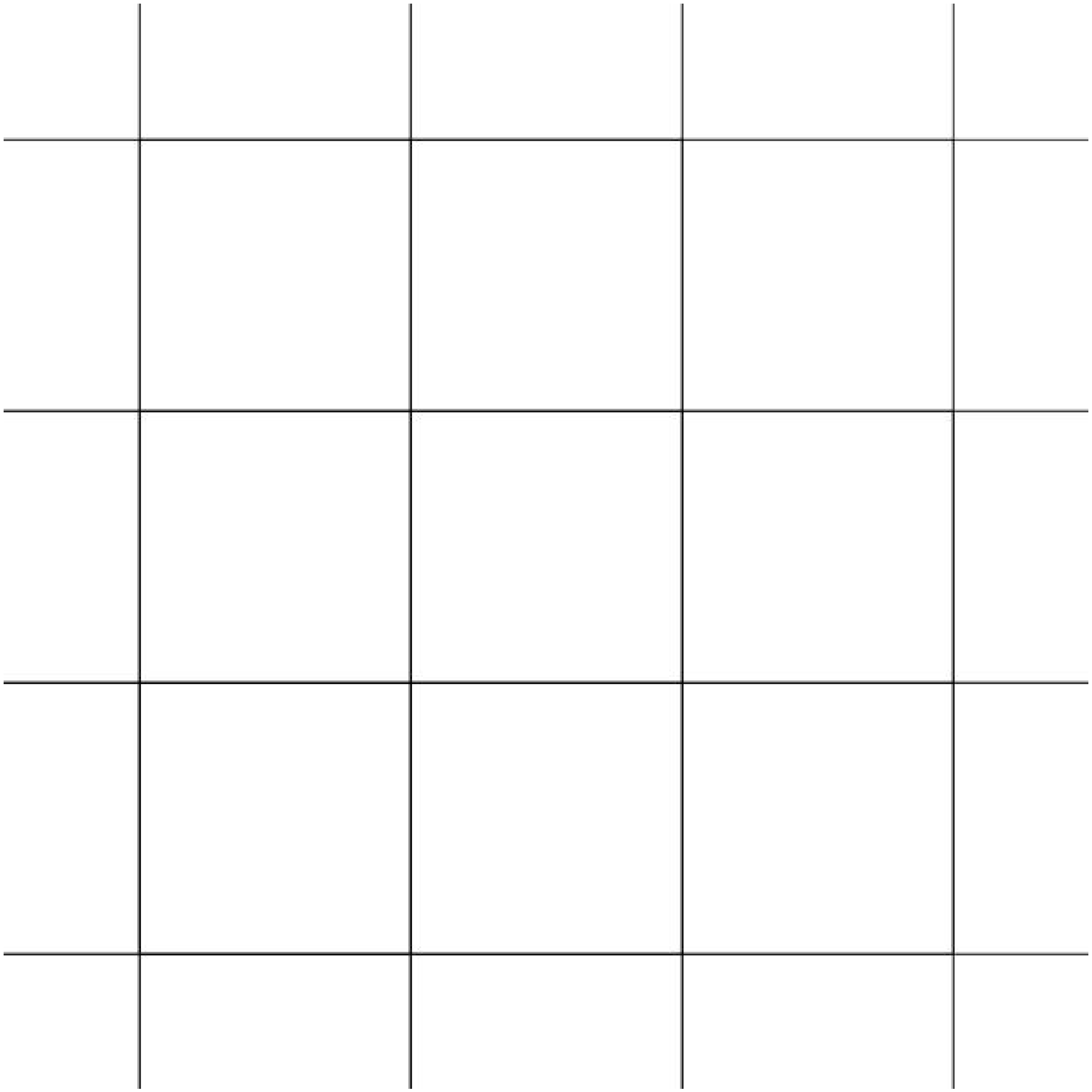


First, second and third generation of tiles.

Condition on angles $\alpha_i = \dfrac{\pi}{m_i}$

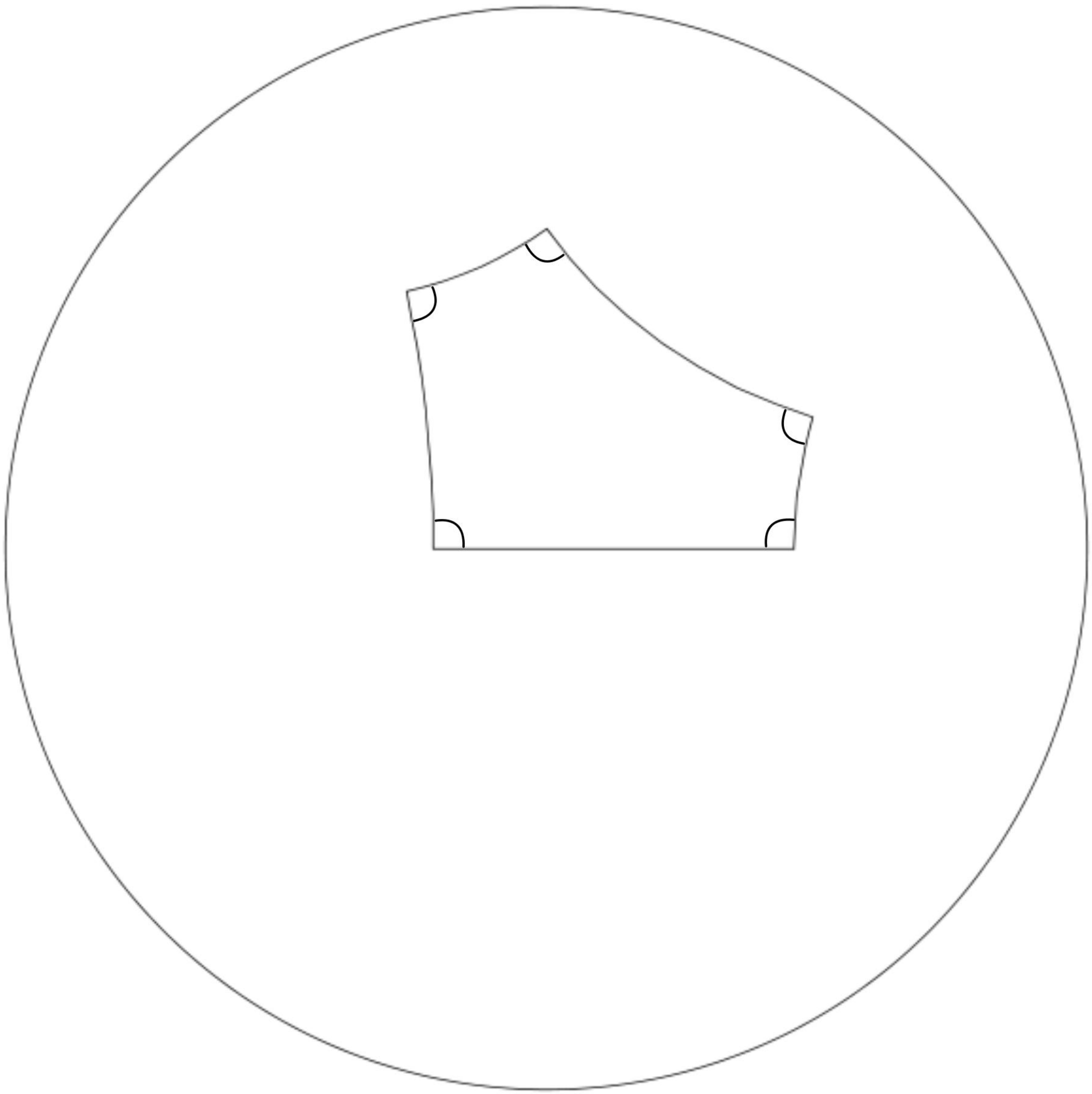
guaranties, that tiles fit without gaps.
$2m_i$ tiles meet at each vertex.

Tiling with Coxeter Polygons



Complete tiling - regular rectangular grid.

Coxeter Polygons in the Hyperbolic Plane

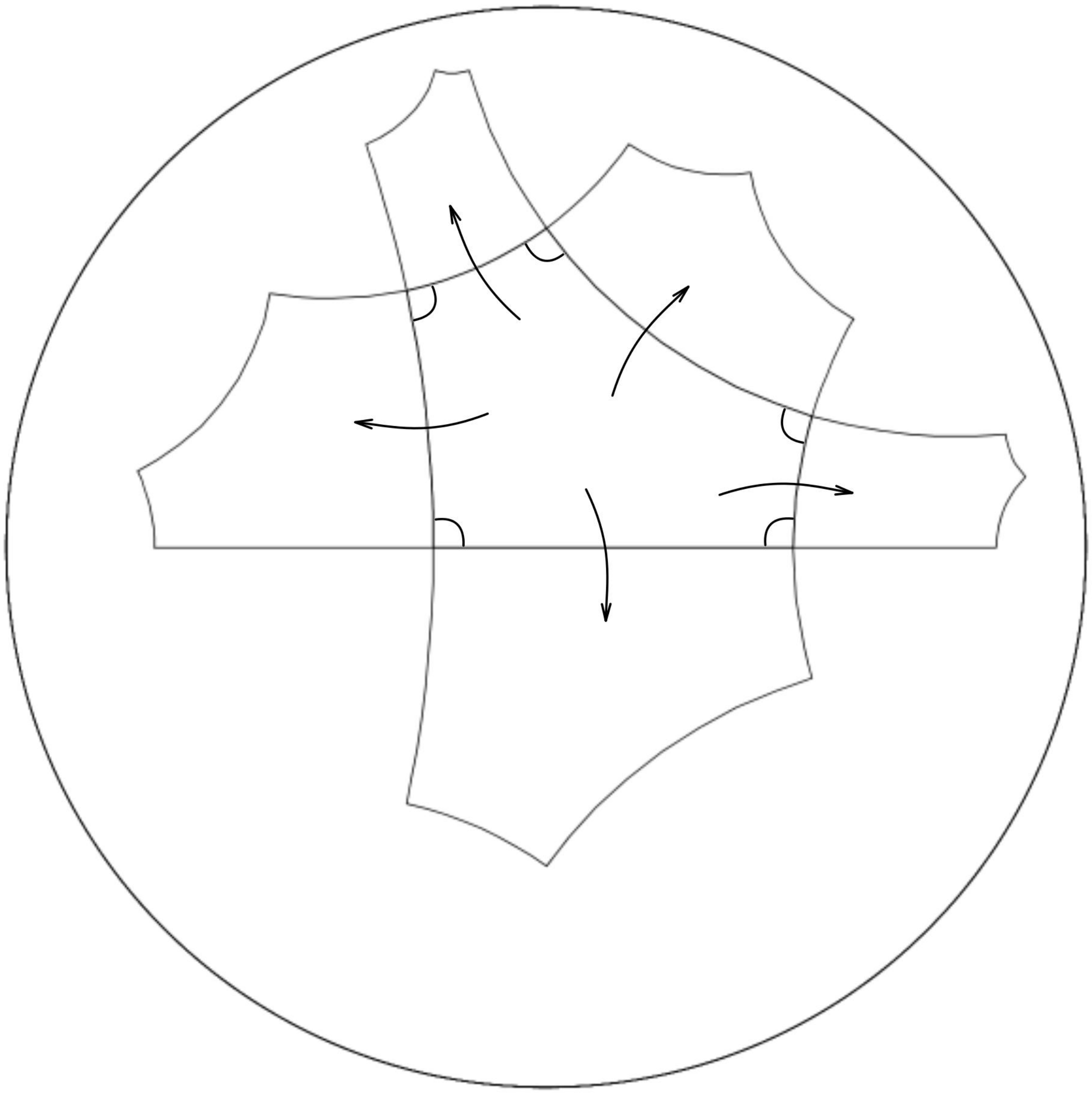


Hyperbolic Coxeter pentagon with all $\alpha_i = \frac{\pi}{2}$ shown in the Poincare disk model of hyperbolic plane.

The shape of the right angled pentagon has 2 independent parameters (two arbitrary sides lengths).

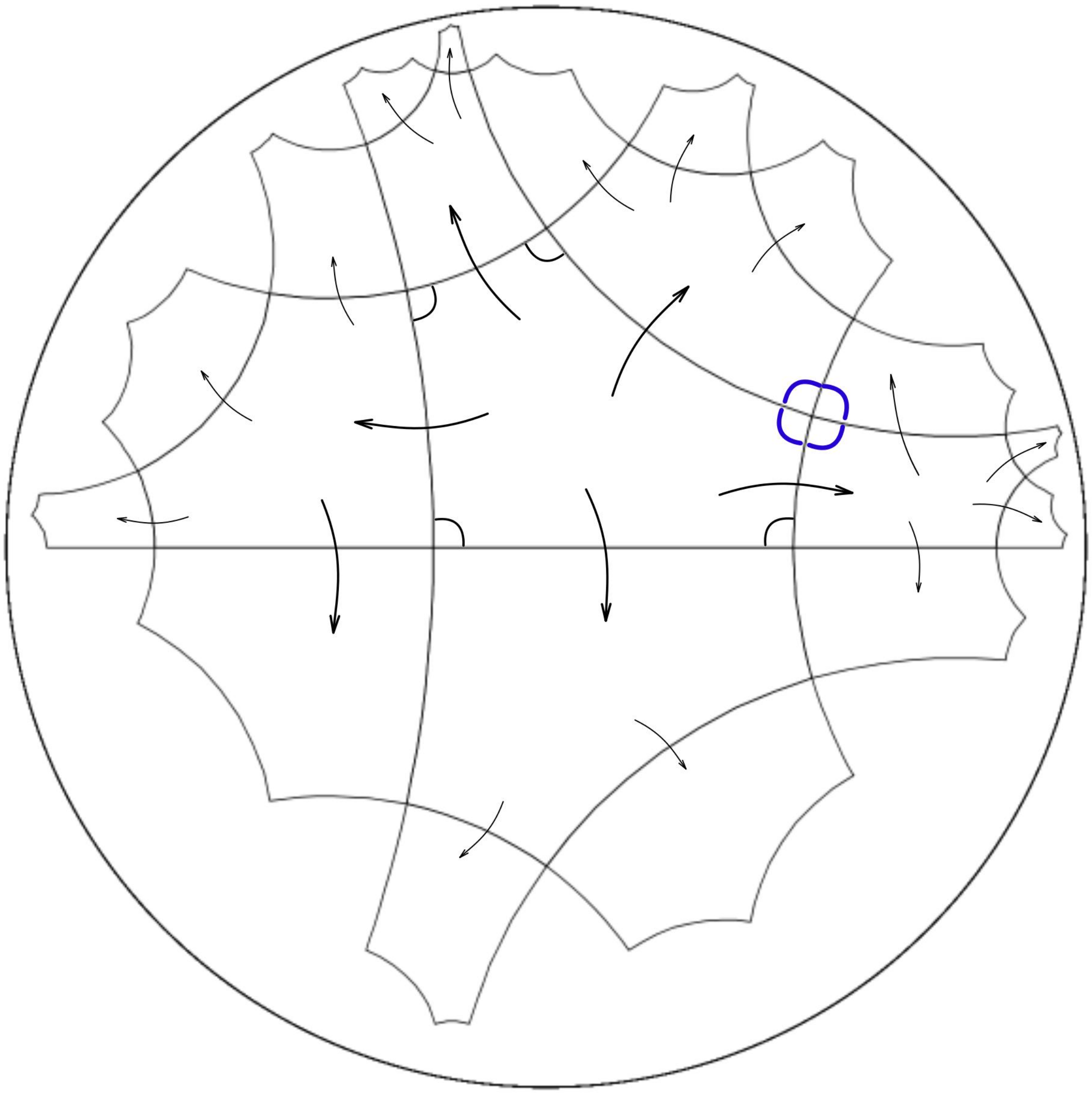
Lengths of any two sides can be chosen independently. The length of three remaining sides can be found from hyperbolic trigonometry identities.

Coxeter Polygons in the Hyperbolic Plane



Base tile is hyperbolically reflected in the sides of the Coxeter polygon.

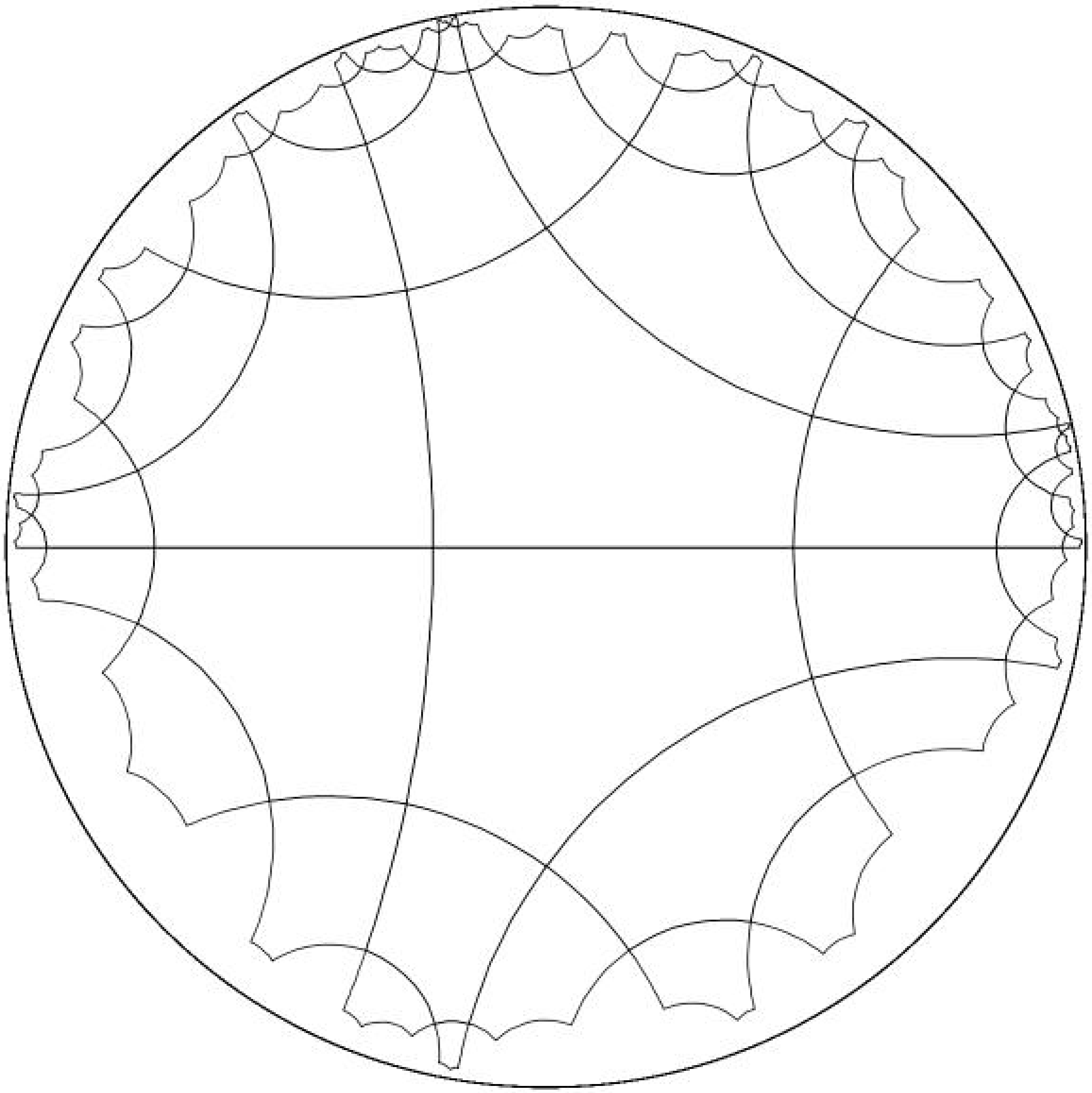
Coxeter Polygons in the Hyperbolic Plane



First, second and third generations of tiles.

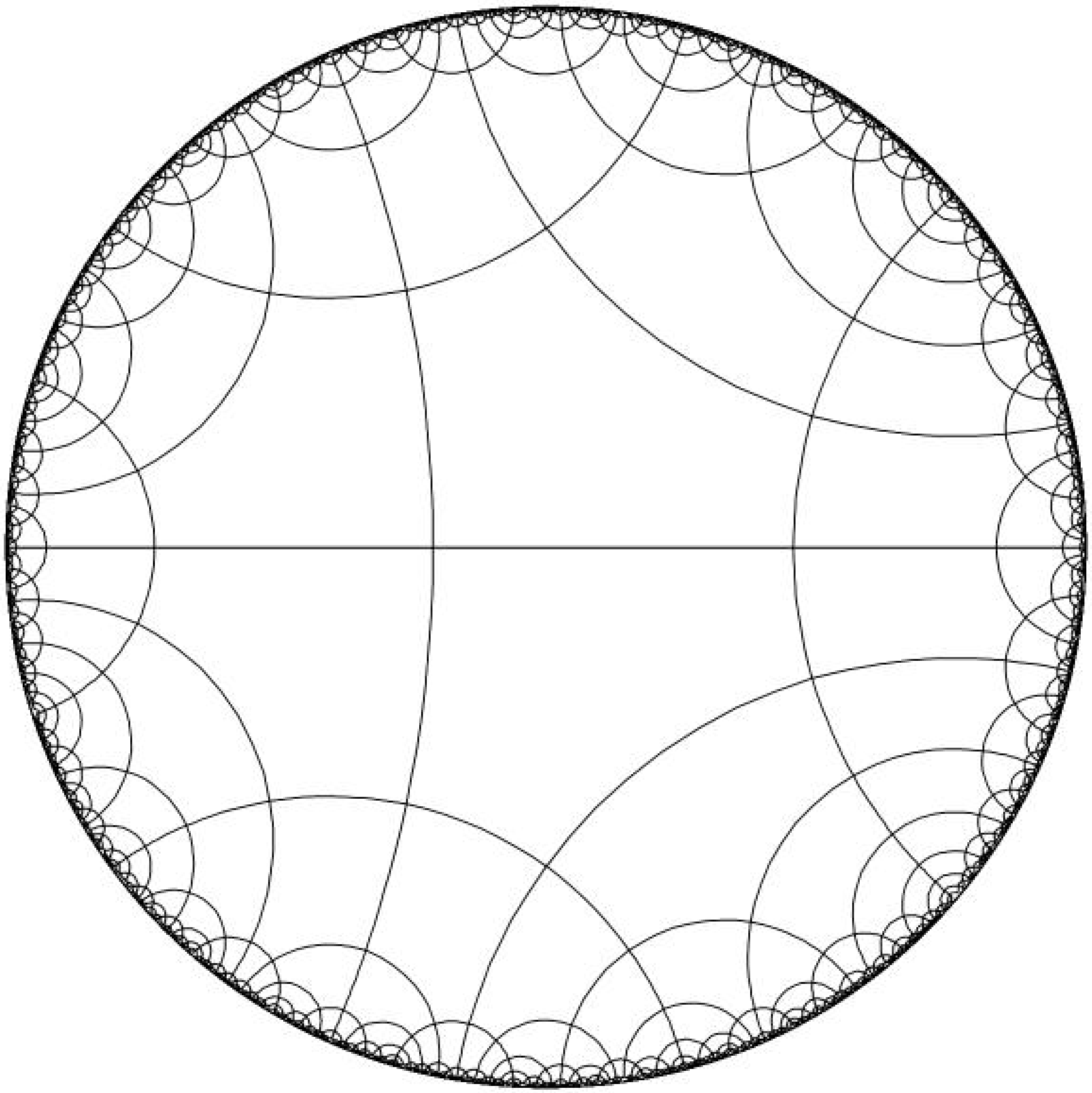
$\alpha_i = \frac{\pi}{m_i}$ guaranties, that $2m_i$ tiles around each vertex fit without gaps and overlaps.

Coxeter Polygons in the Hyperbolic Plane



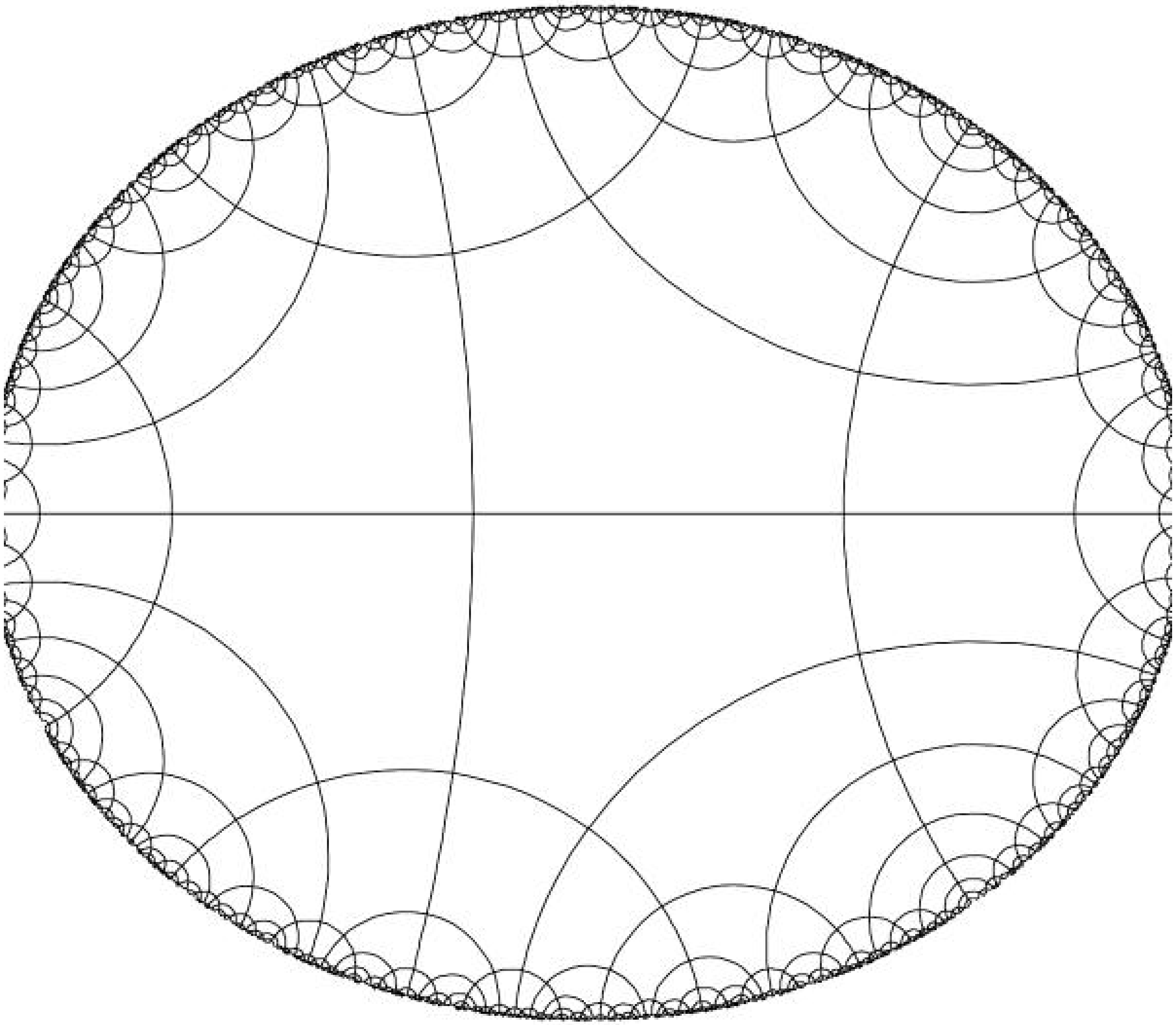
Four generations of tiles.

Coxeter tiles in the Hyperbolic Plane



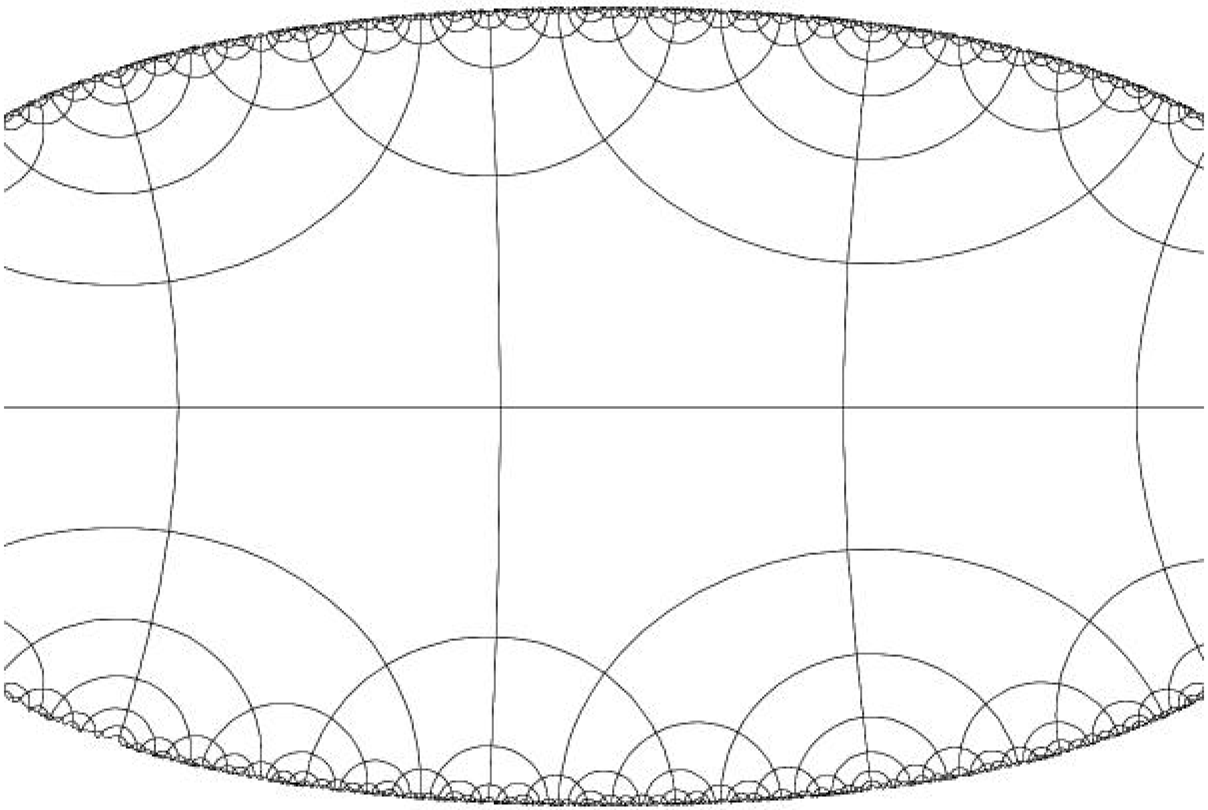
Complete tiling in the Poincare disc model of hyperbolic plane.

Coxeter tiles in the band model



To make another view of the same tiling we can conformally stretch the Poincare disk into an infinite band.

Coxeter tiles in the band model



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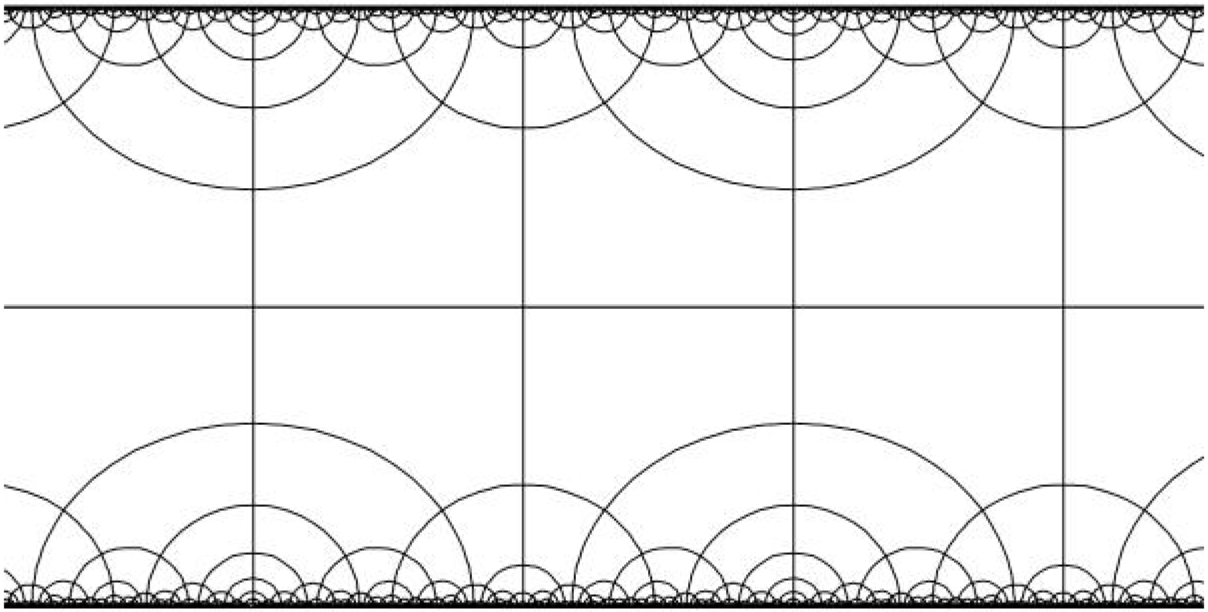
Coxeter tiles in the band model

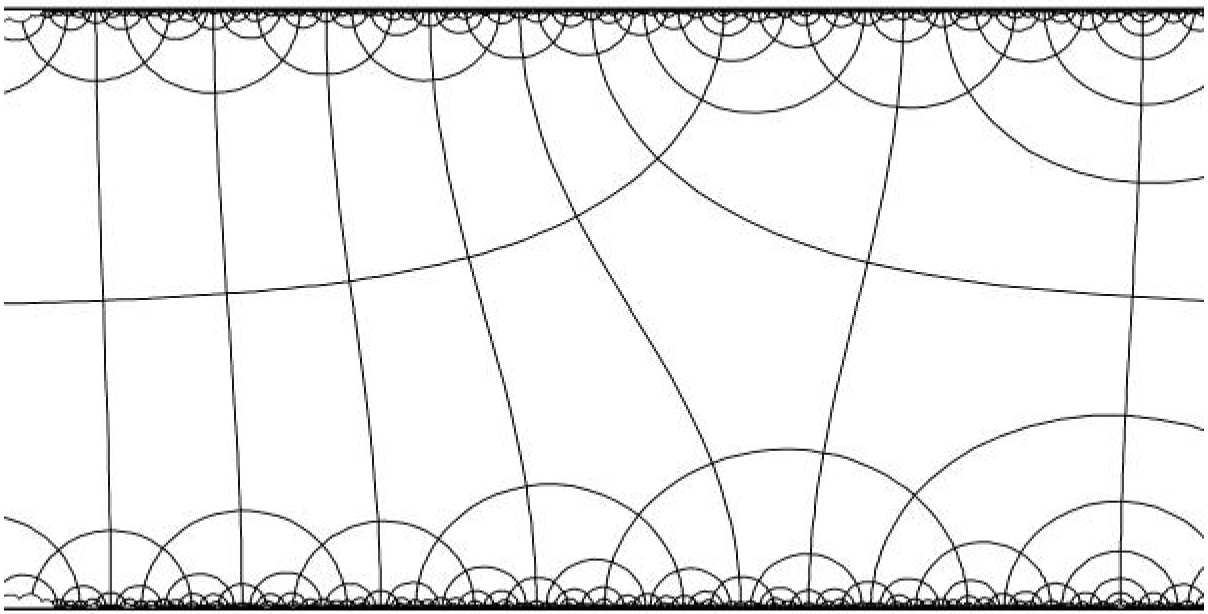


Tiling in the band model of the hyperbolic plane.

The hyperbolic metric along horizontal axis of the band is euclidean. As a result the tiling has euclidean translation symmetry in horizontal direction.

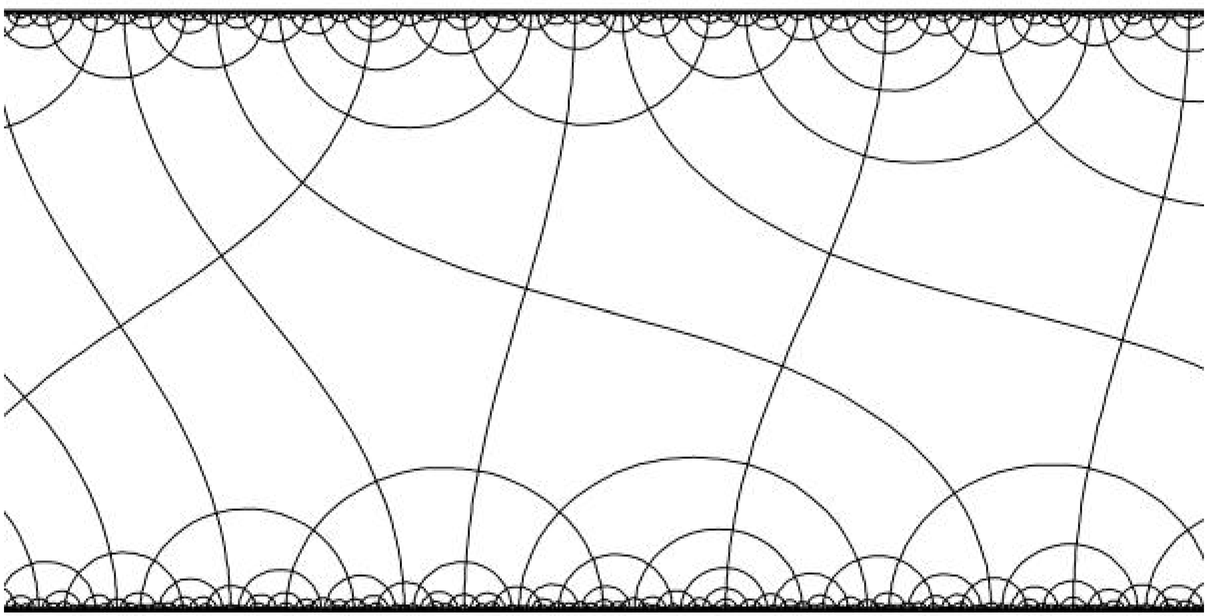
The tiling has translation symmetry along each of the geodesics shown, but only horizontal translation is obvious to our euclidean eyes.

Motion in the band model



Appropriate hyperbolic isometry can send any 2 selected points to $\pm\infty$ of the band model.

Motion in the band model



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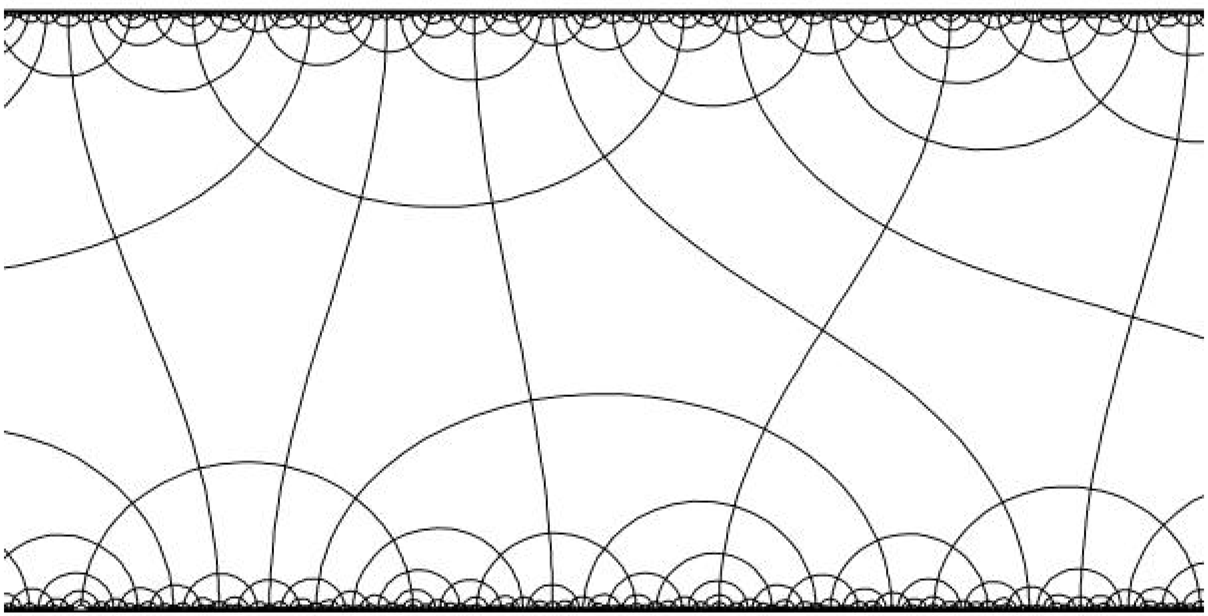
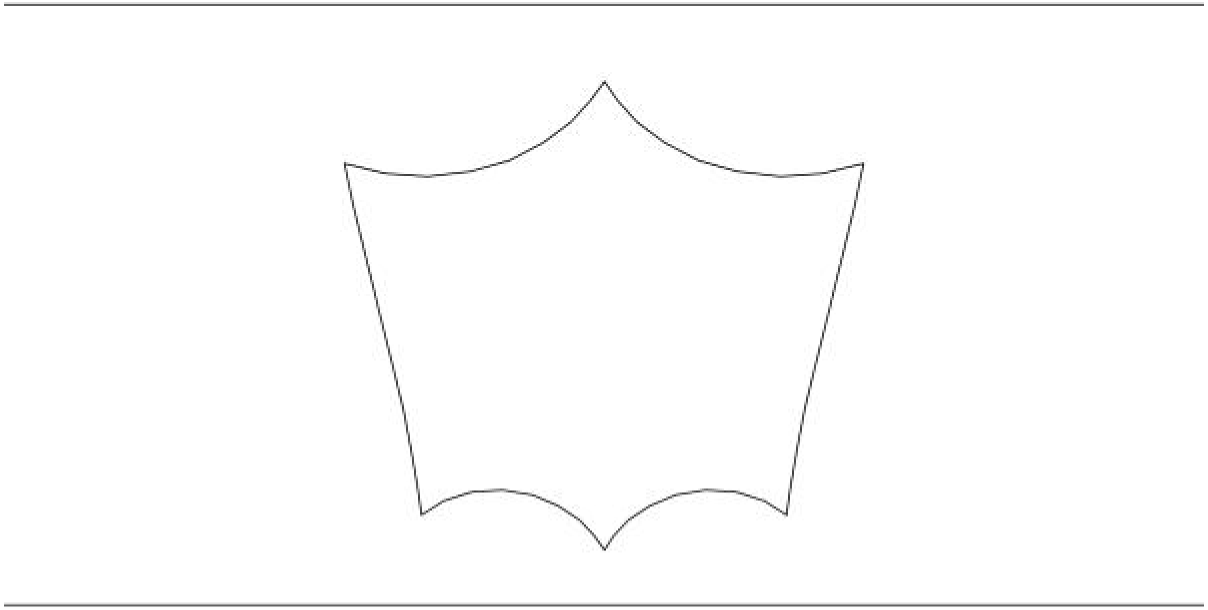
Motion in the band model



Appropriate hyperbolic isometry can send any 2 selected points to $\pm\infty$ of the band model.

Existence of Coxeter Polygons

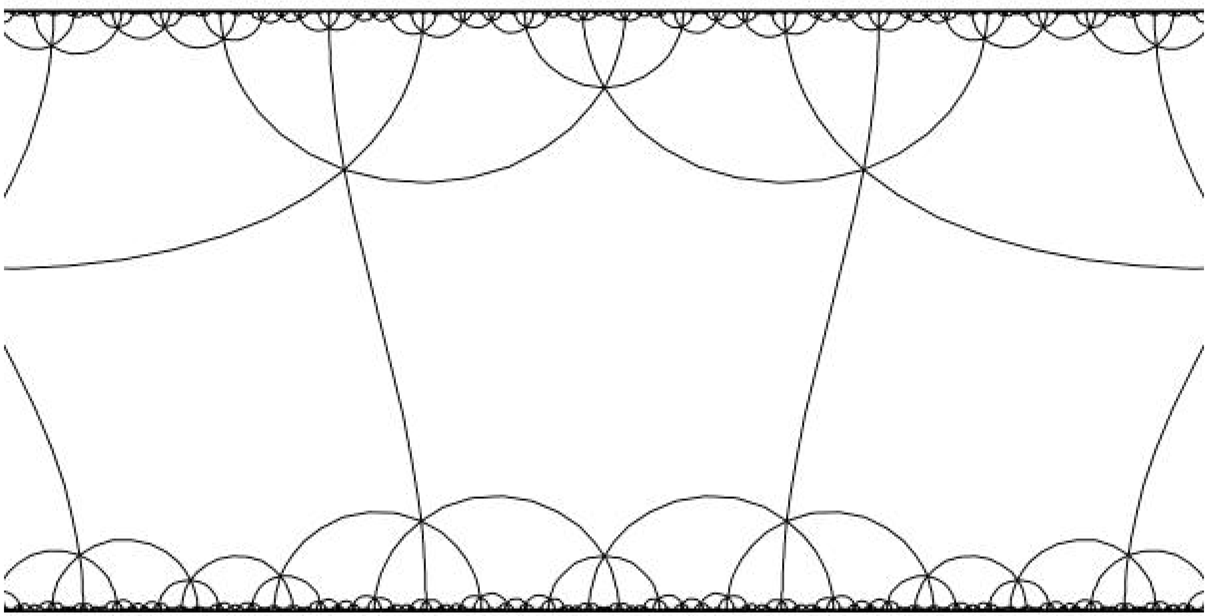


Hyperbolic Coxeter N-gon exist for any $N \geq 5$ and angles $\alpha_i = \dfrac{\pi}{m_i}$ $m_i \geq 2$

Hyperbolic triangle exist if
$$\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} < 1$$
Hyperbolic quadrangle exist if
$$\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} + \frac{1}{m_4} < 2.$$

The space of shapes of Coxeter N-gon has dimension $N - 3$.

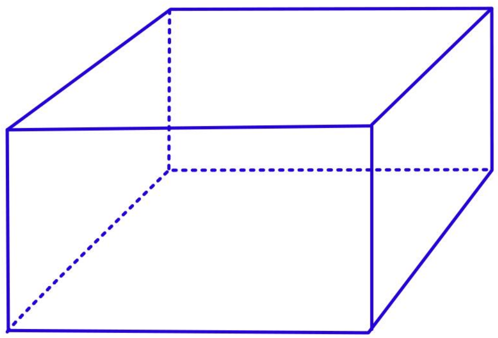
Existence of Coxeter Polygons



Example.

Tiling of hyperbolic plane with Coxeter hexagons with all angles equal $\frac{\pi}{3}$.

Coxeter Polyhedron



Dihedral angles of *Coxeter polyhedron* are submultiples of π:

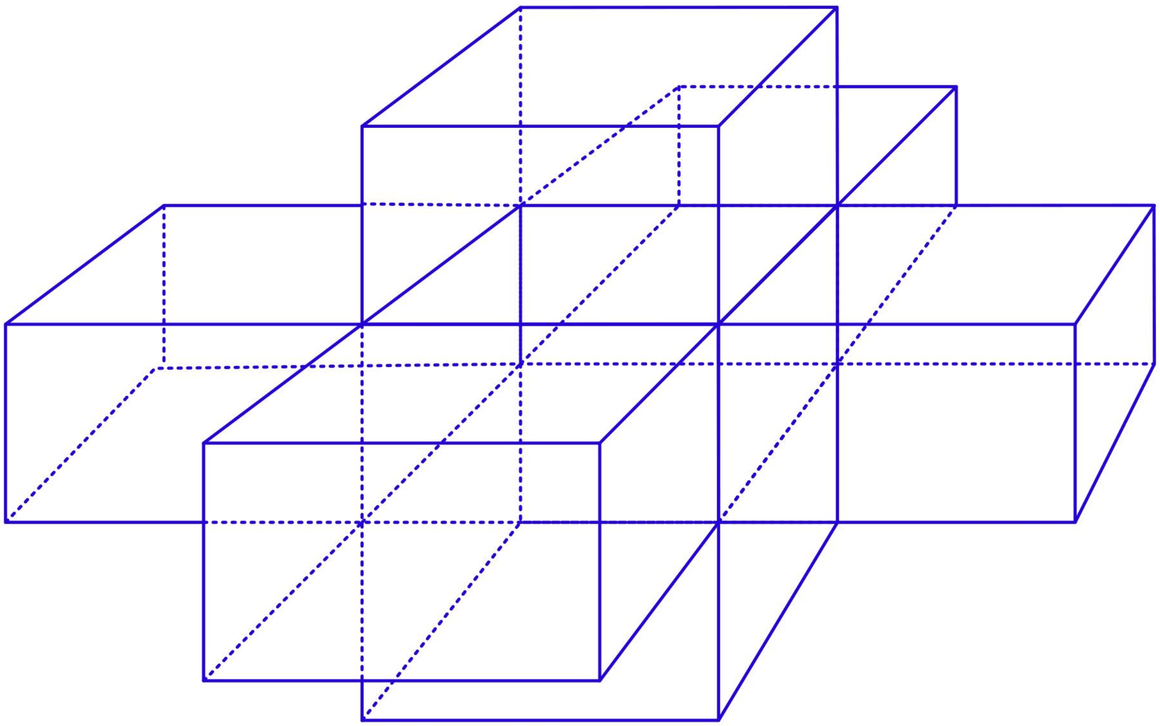
$$\delta_i = \frac{\pi}{m_i}.$$

Reflections in sides of Coxeter polyhedron satisfy conditions of *Poincare Polyhedron Theorem*.

Therefore the reflected copies of such polyhedron fill the space without gaps and overlaps.

Example - euclidean rectangular parallelepiped.

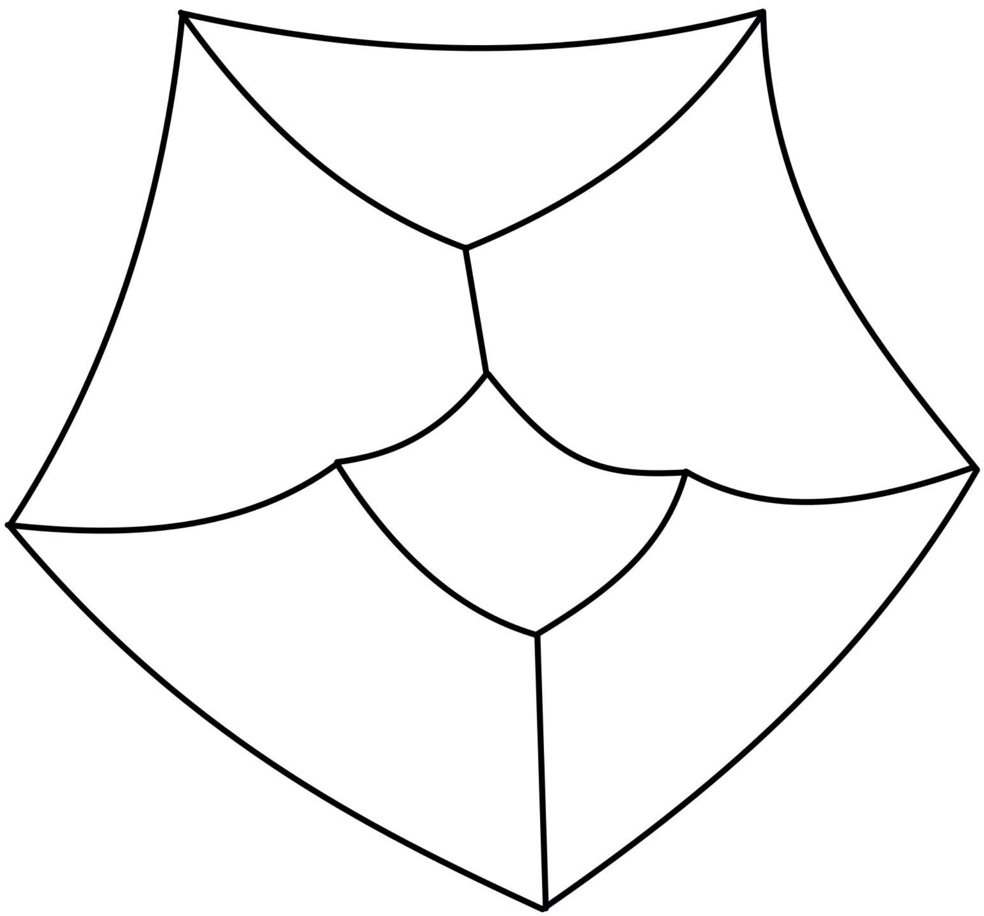
Coxeter Polyhedra Tiling



First and second generation of tiling of the euclidean space by rectangular parallelepipeds.

We can easy imagine the rest of the tiling - infinite regular grid.

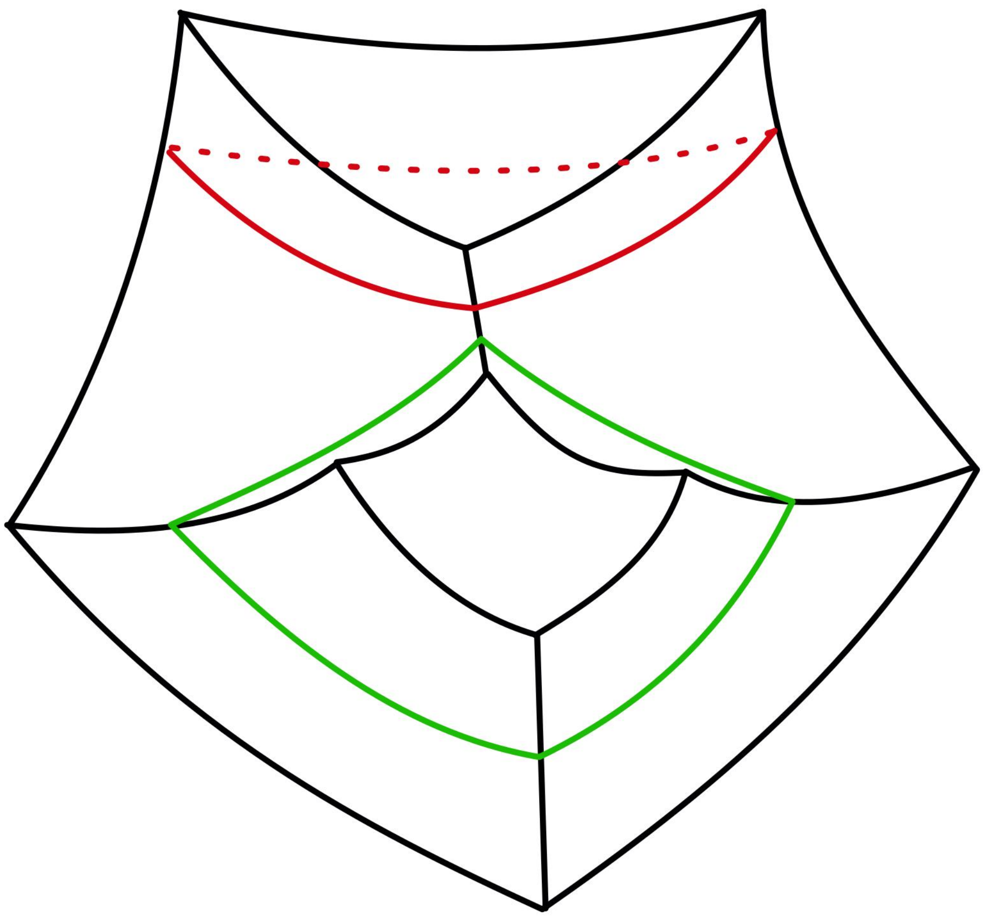
Coxeter polyhedra in H^3



Coxeter polyhedra in hyperbolic space.

Do they exist?

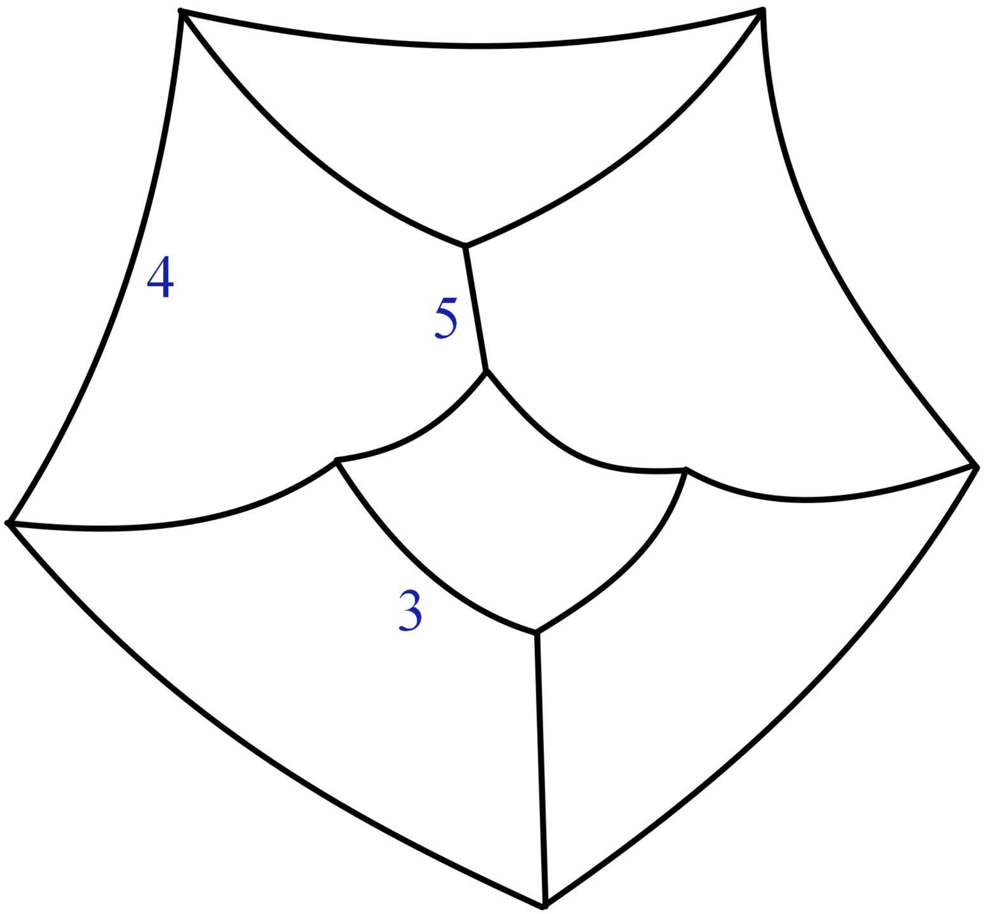
Coxeter polyhedra in H^3



From Andreev's Theorem(1967) on compact polyhedra with non-optuse dihedral angles in hyperbolic space follows:

There exist unique compact Coxeter polyhedron in hyperbolic space iff:
1) Each vertex has 3 adjacent faces.
2) Tripple of dihedral angles at each vertex is from the set
$\left(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}\right), \left(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}\right), \left(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{5}\right),$
$\left(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{n}\right), n \geq 2,$
3) Dihedral angles for each prismatic 3-cycle satisfy $\delta_1 + \delta_2 + \delta_3 < \pi$
4) Dihedral angles for each prismatic 4-cycle satisfy $\delta_1 + \delta_2 + \delta_3 + \delta_4 < 2\pi$

Coxeter polyhedra in H^3

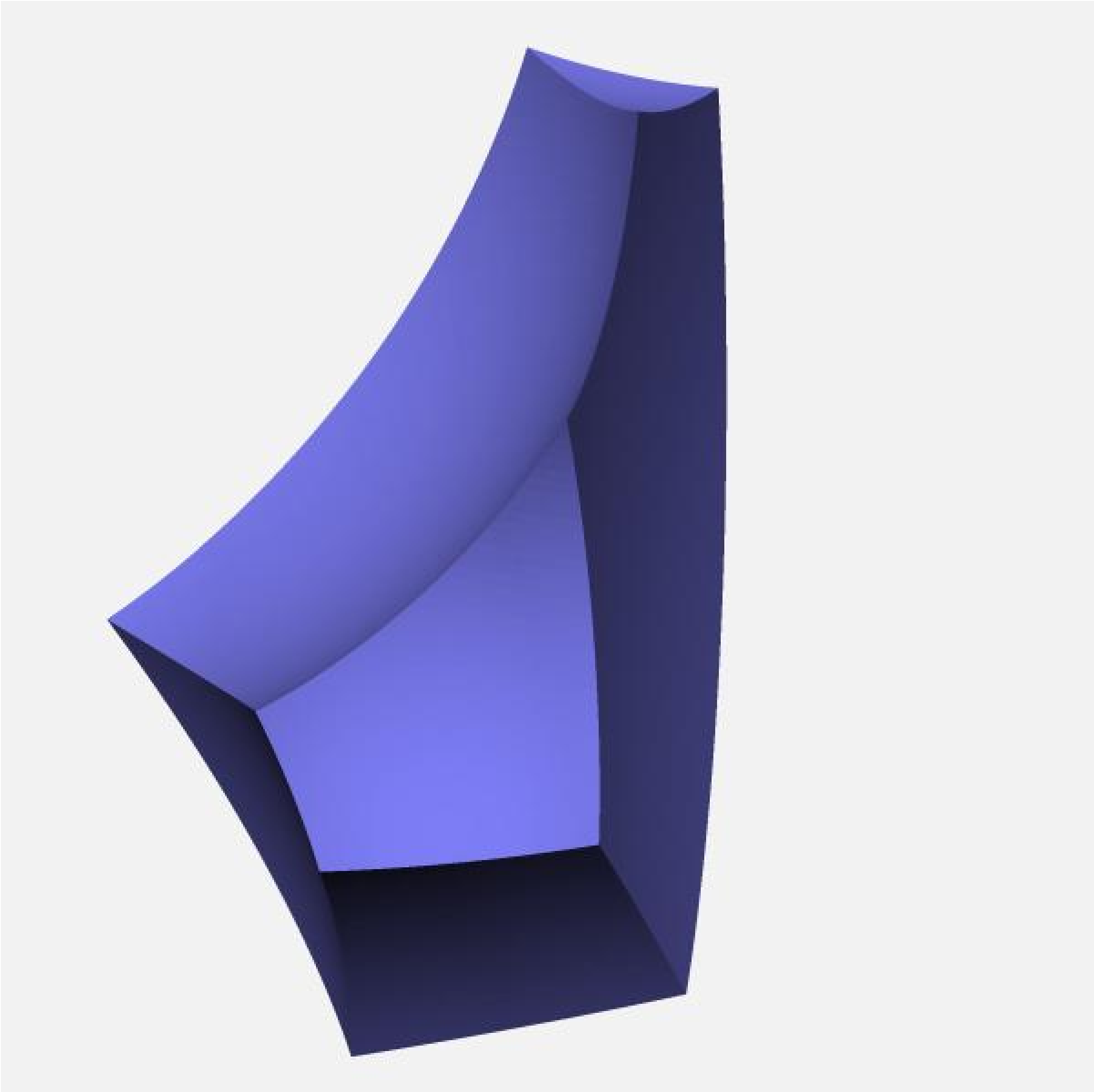


Example of polyhedron satisfying Andreev's theorem.

Edges with dihedral angles $\delta = \frac{\pi}{m}$ are marked with m if $m > 2$
Unmarked edges have dihedral angles $\frac{\pi}{2}$.

The polyhedron is combinatorially equivalent to a cube with one truncated vertex.

Example. Truncated cube in H^3



Visualization of the actual geometric realization of truncated cube in the hyperbolic space shown in the Poincare ball model.

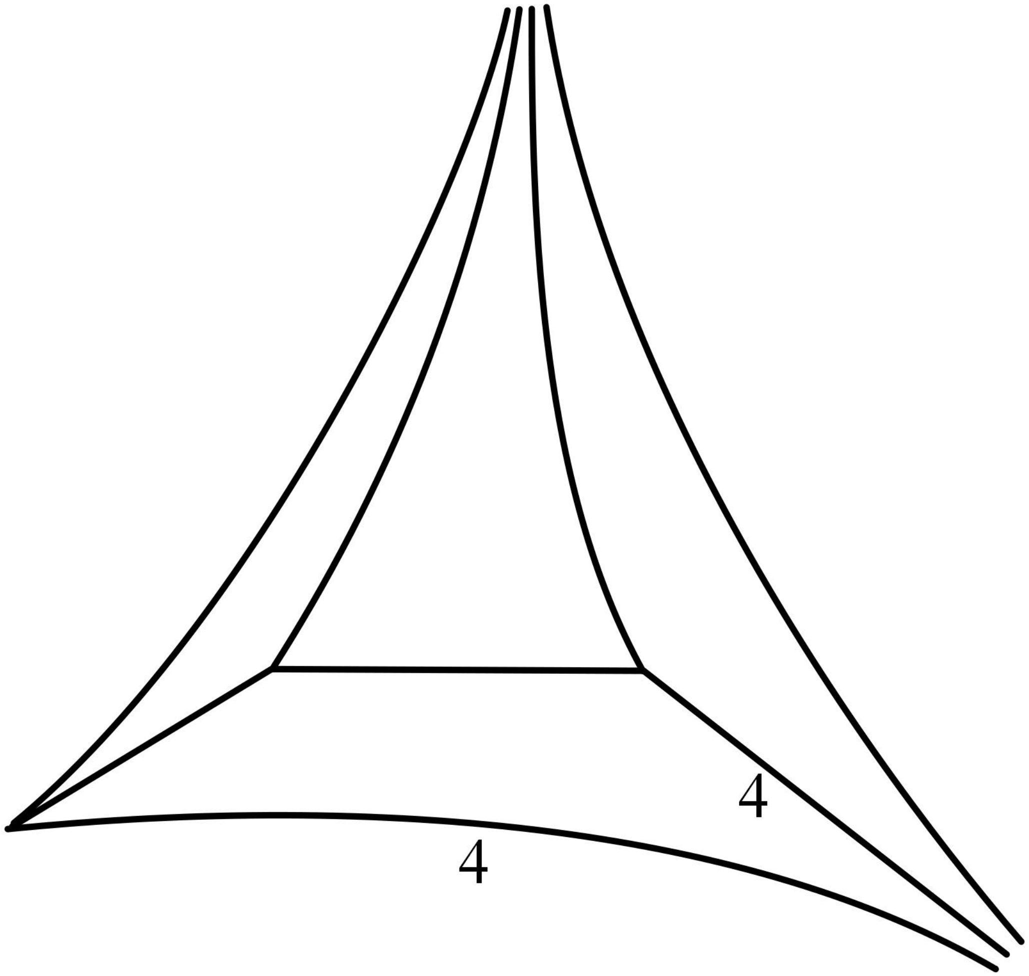
Example. Right angled dodecahedron

Regular right angled hyperbolic dodecahedron

All dihedral angles are equal $\frac{\pi}{2}$

All faces are regular right angled hyperbolic pentagons

Non-compact Coxeter polyhedra in H^3



From Andreev's Theorem(1967) on polyhedra of finite volume with non-optuse dihedral angles in hyperbolic space follows:

There exist unique Coxeter polyhedron in H^3 if

1) the additional triplets of dihedral angles are allowed:

$\left(\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4}\right), \left(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}\right), \left(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{6}\right)$

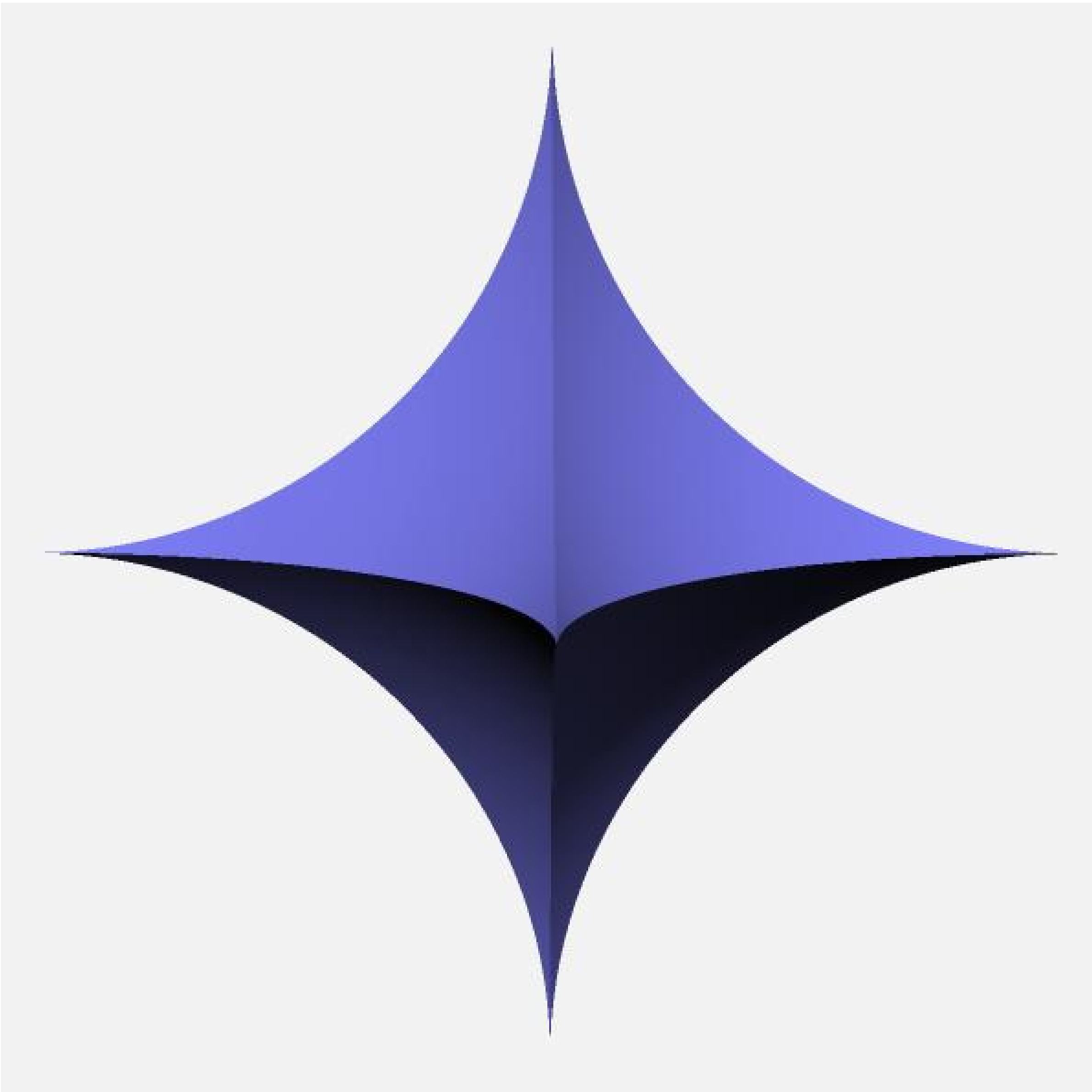
Vertices with such angles are located at the infinity of hyperbolic space (ideal vertices).

2) Ideal vertices with 4 adjacent faces and all four dihedral angles equal $\frac{\pi}{2}$ are permitted.

Example. Ideal hyperbolic tetrahedron

Tetrahedron with all dihedral angles $\frac{\pi}{3}$. All 4 vertices are ideal.

Example. Ideal hyperbolic octahedron

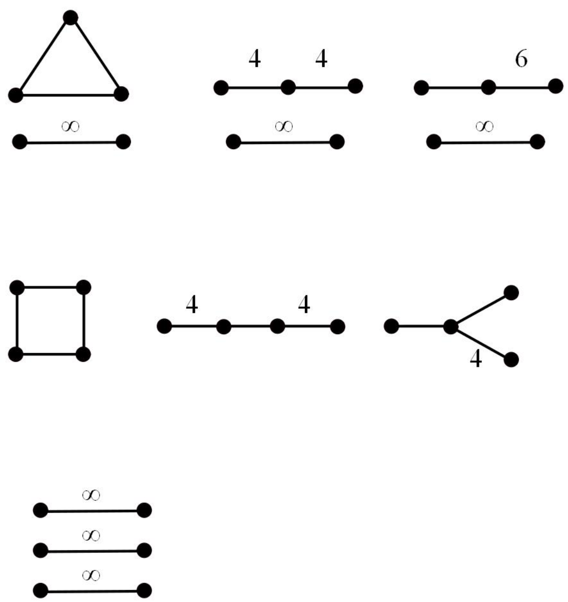


Octahedron with all dihedral angles $\frac{\pi}{2}$. All 6 vertices are ideal.

Coxeter polyhedra in E^3 vs H^3

Coxeter polyhedra in Euclidean space

Only 7 non-equivalent types of compact polyhedra:



Coxeter polyhedra in Hyperbolic space

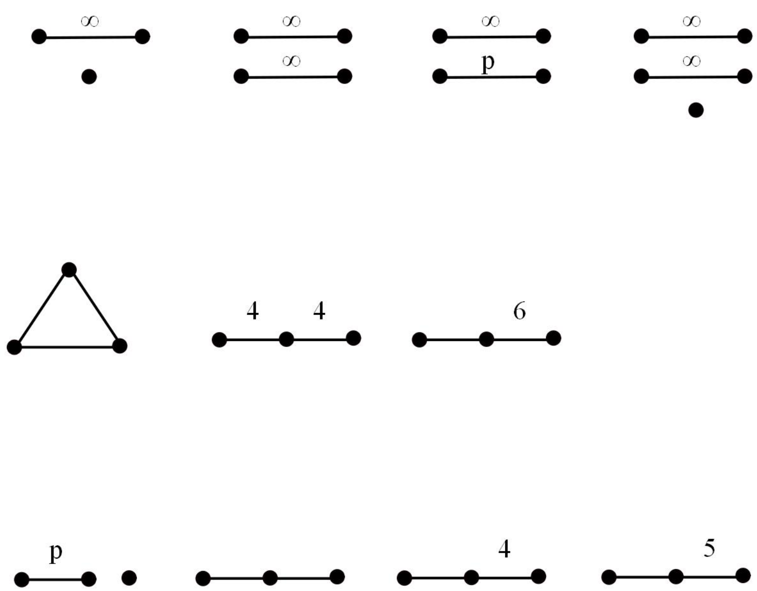
- Infinitely many combinatorially non-equivalent compact polyhedra.
- Each combinatorial type allows wide selection of different dihedral angles satisfying Andreev's theorem.

Coxeter polyhedra in E^3 vs H^3 (continued)

Coxeter polyhedra in Euclidean space

- Shape may have continuous parameters, for example height of a prism.

- Few families of infinite polyhedra

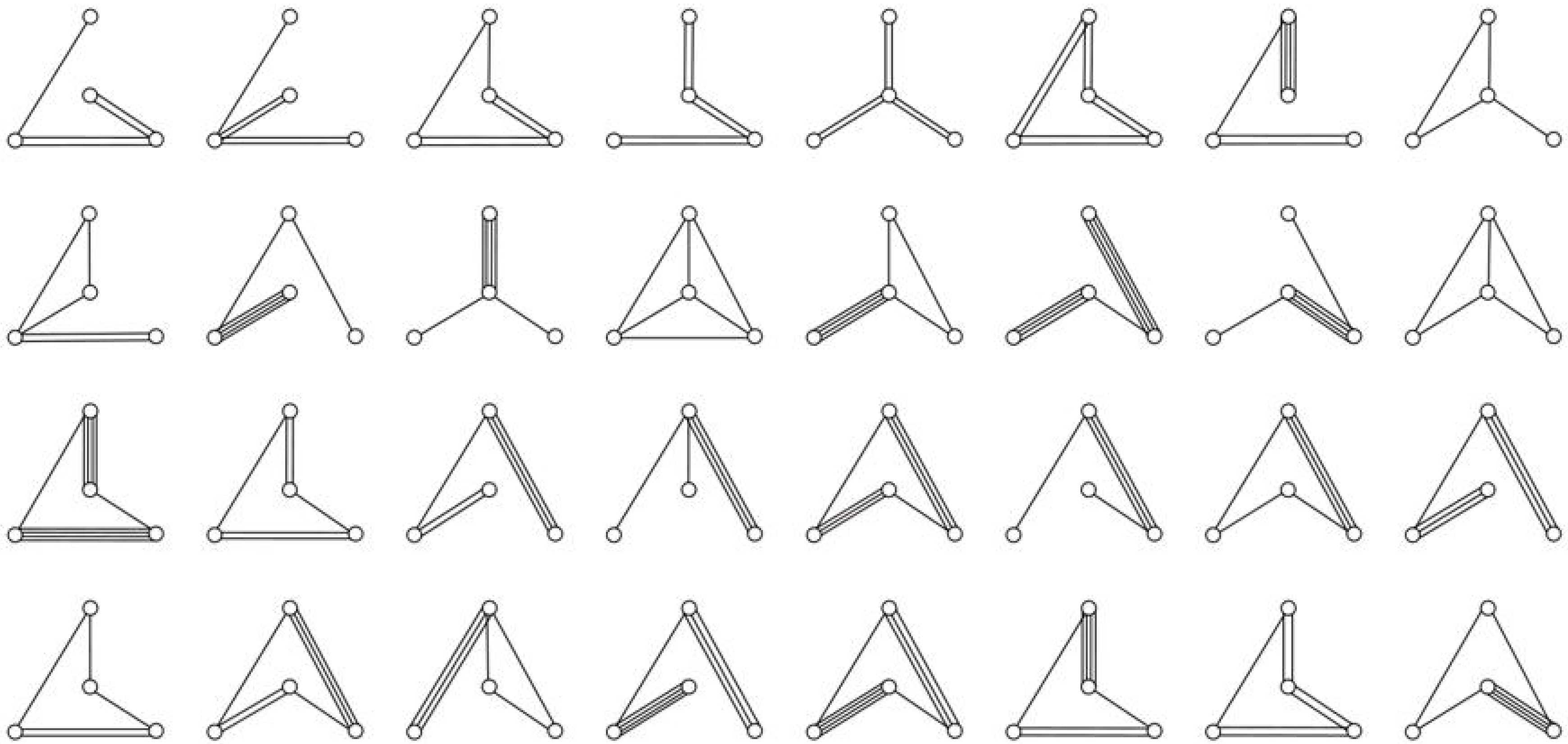


Coxeter polyhedra in Hyperbolic space

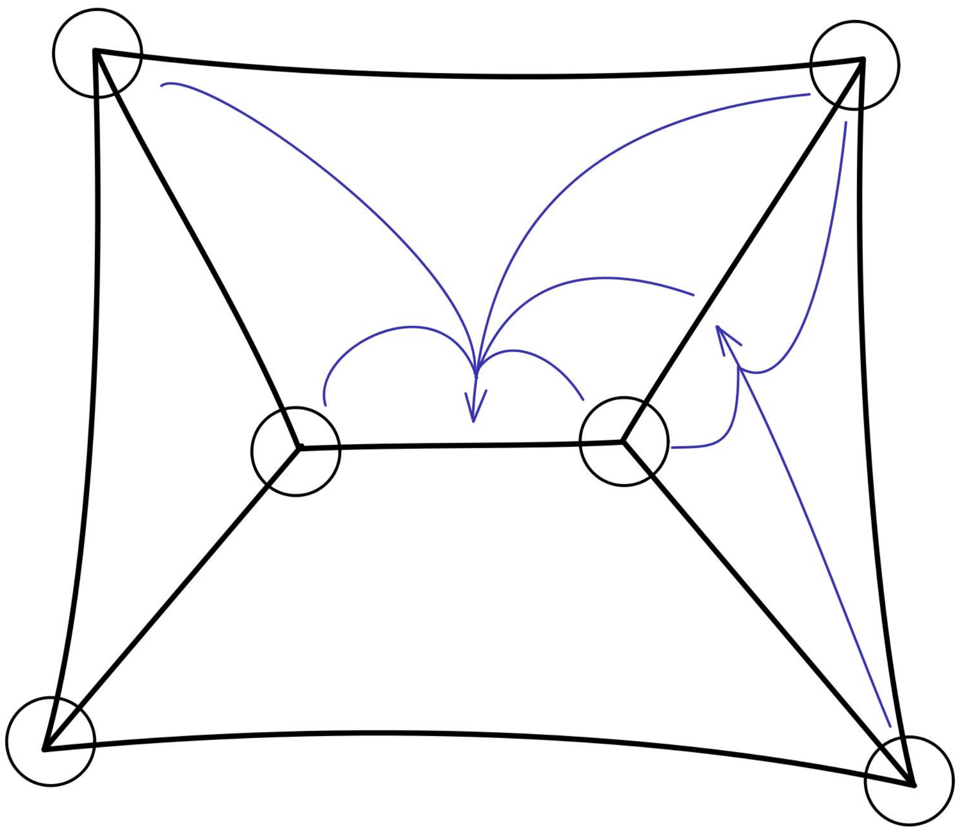
- Shape of polyhedron of finite volume is fixed by the choice of it's dihedral angles.

- Infinite number of infinite families of infinite polyhedra of finite and infinite volume.

Example: 32 Coxeter tetrahedra in H^3



How to construct polyhedra in H^3



"Hand" calculation for simpler polyhedra:

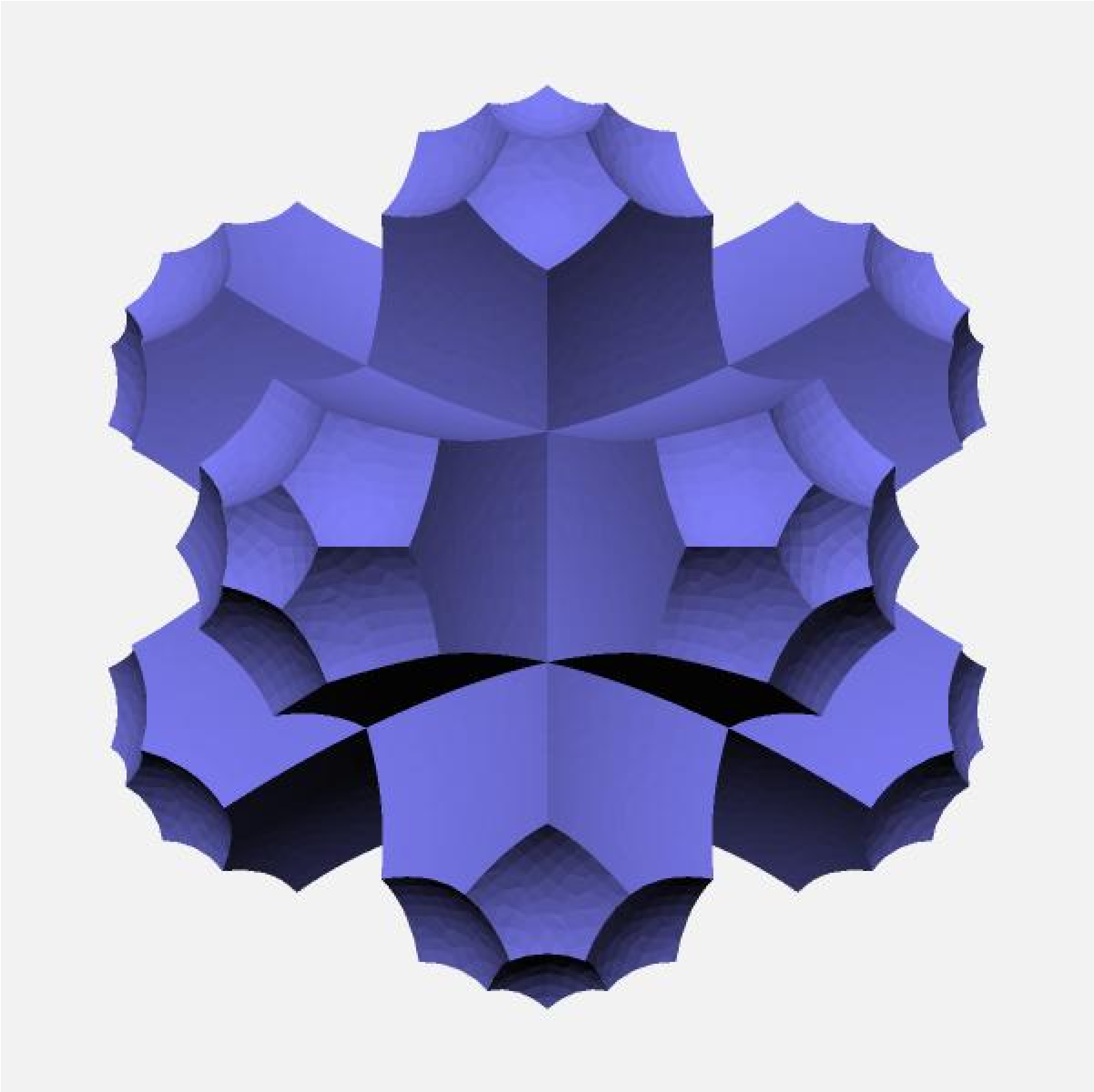
- Construct small sphere around each vertex. Intersections of adjacent faces with the spheres form spherical triangles with angles equal to dihedral angles.
- Using spherical sines laws find sides of spherical triangles from angles. These sides are flat angles of polyhedron's faces.
- From known flat angles of triangular faces find their edges using hyperbolic sines laws.
- From known angles and some edges find remaining edges using hyperbolic trigonometry.

General method: Roeder R. (2007) Constructing Hyperbolic Polyhedra Using Newton's Method.

Visualization of tiling in H^3

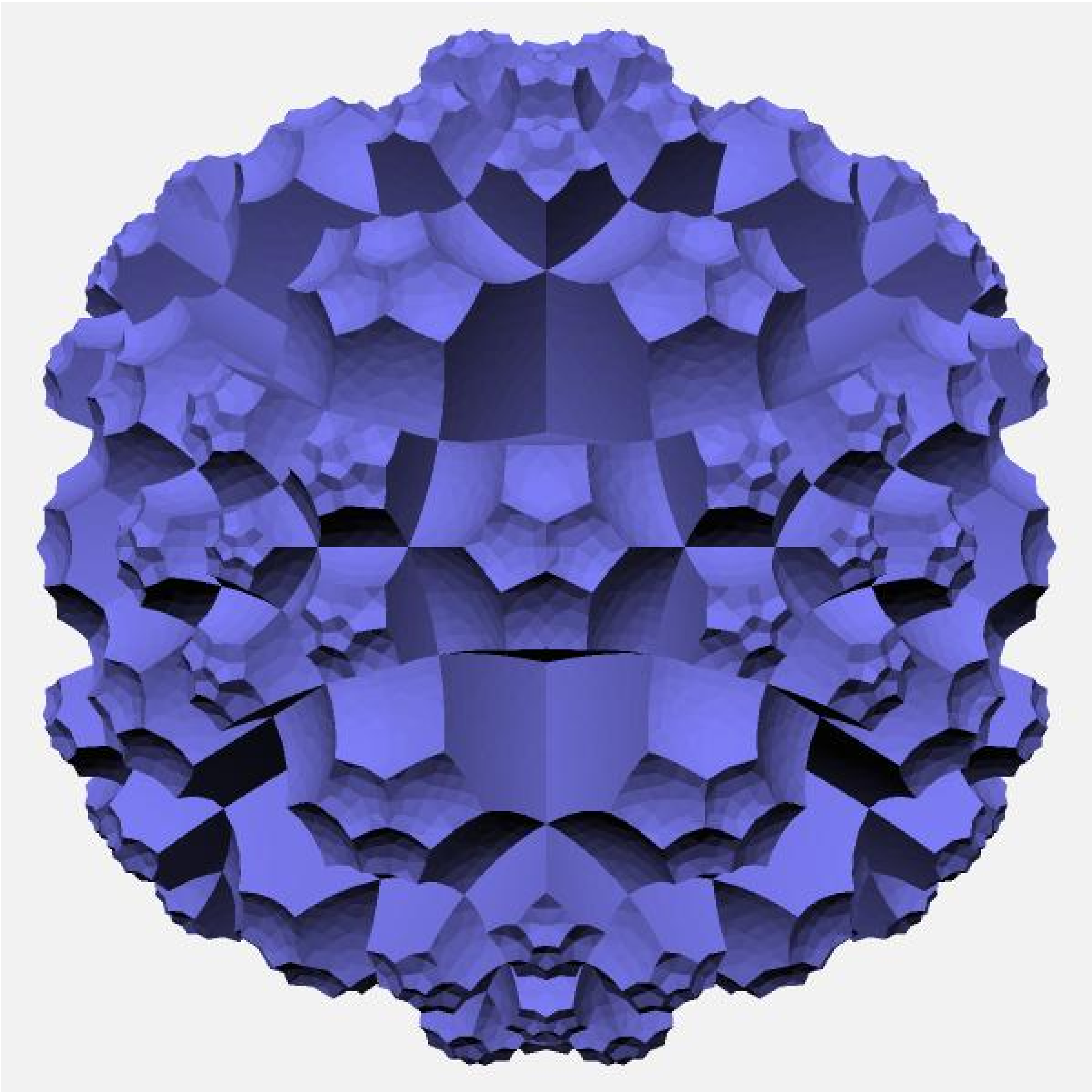
Regular right angled dodecahedron.
12 generators of the tiling are reflections in each of 12 dodecahedron's faces

Visualization of tiling in H^3



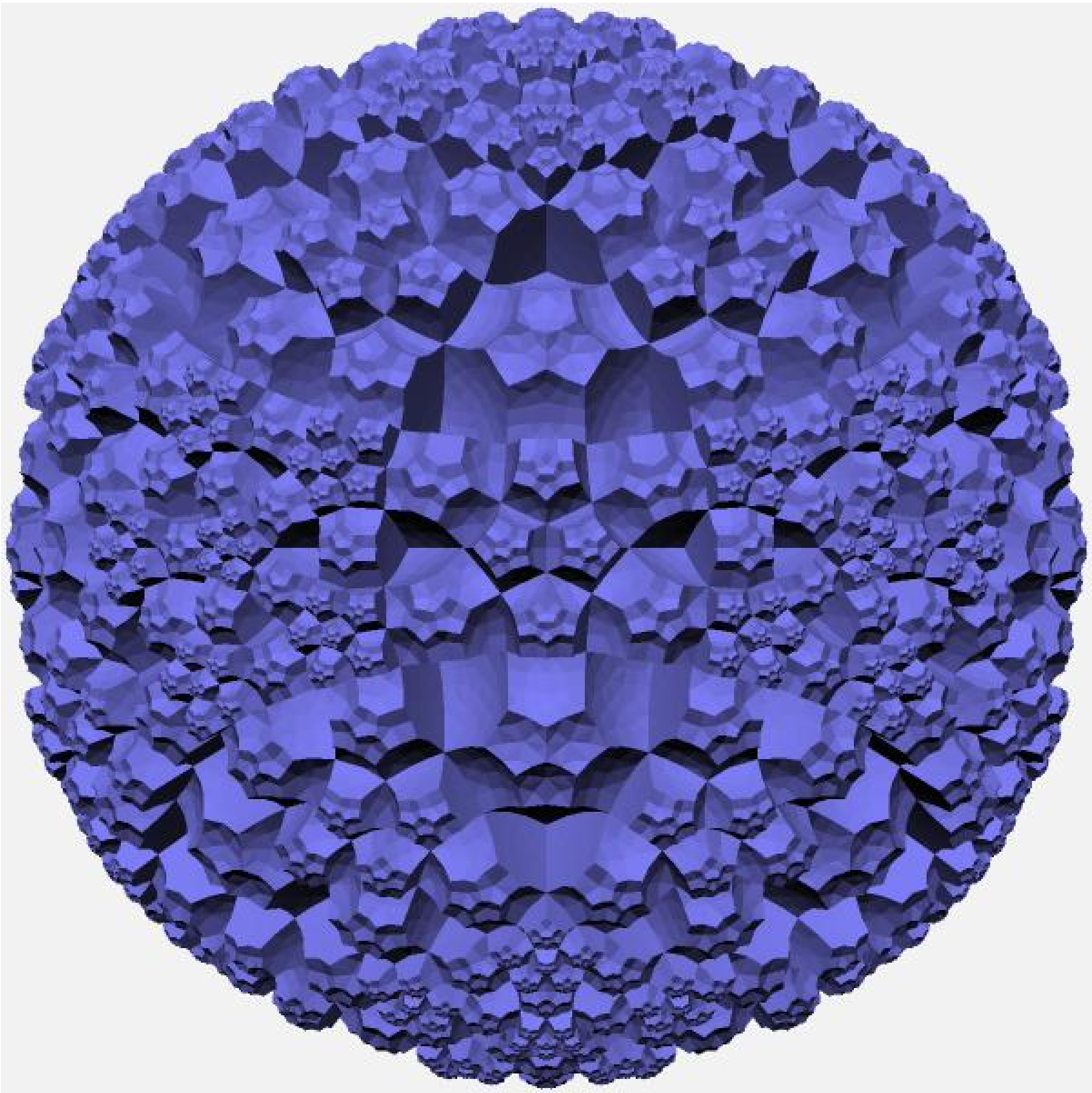
Regular right angled dodecahedron.
First iteration of reflections.

Visualization of tiling in H^3



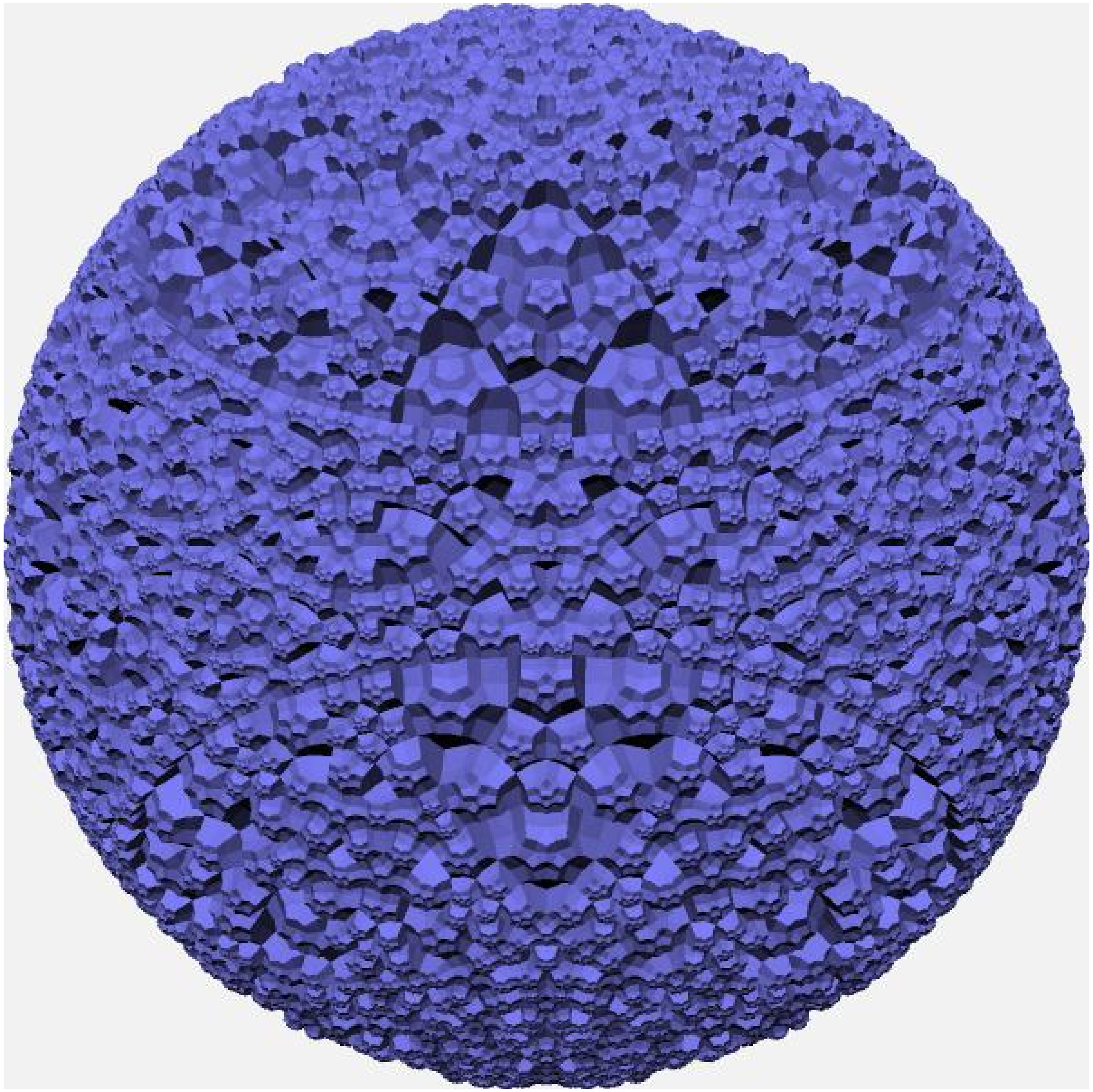
Regular right angled dodecahedron.
2nd iteration.

Visualization of tiling in H^3



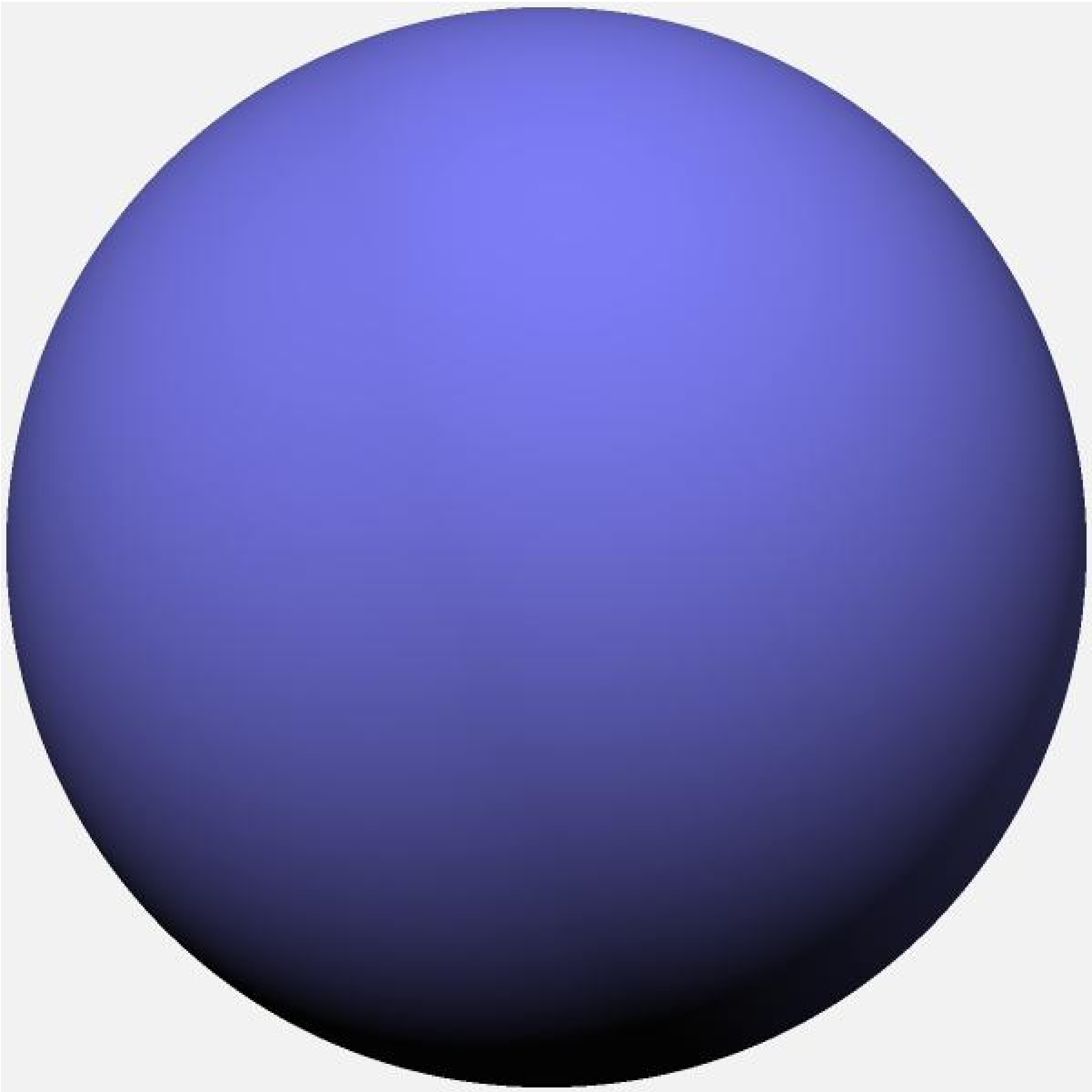
Regular right angled dodecahedron.
3rd iteration.

Visualization of tiling in H^3



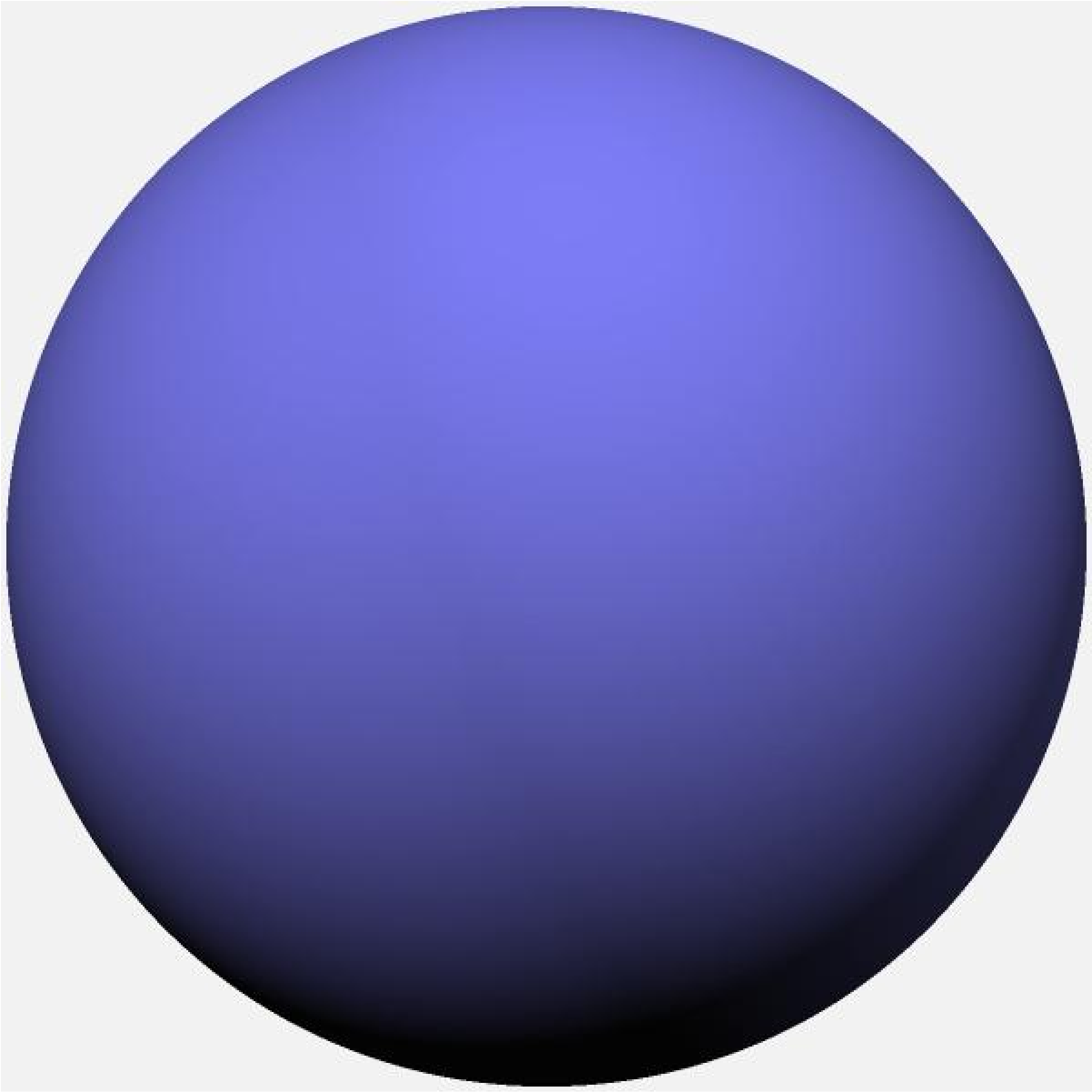
Regular right angled dodecahedron.
4th iteration.

Visualization of tiling in H^3



Regular right angled dodecahedron.
10-th iteration.

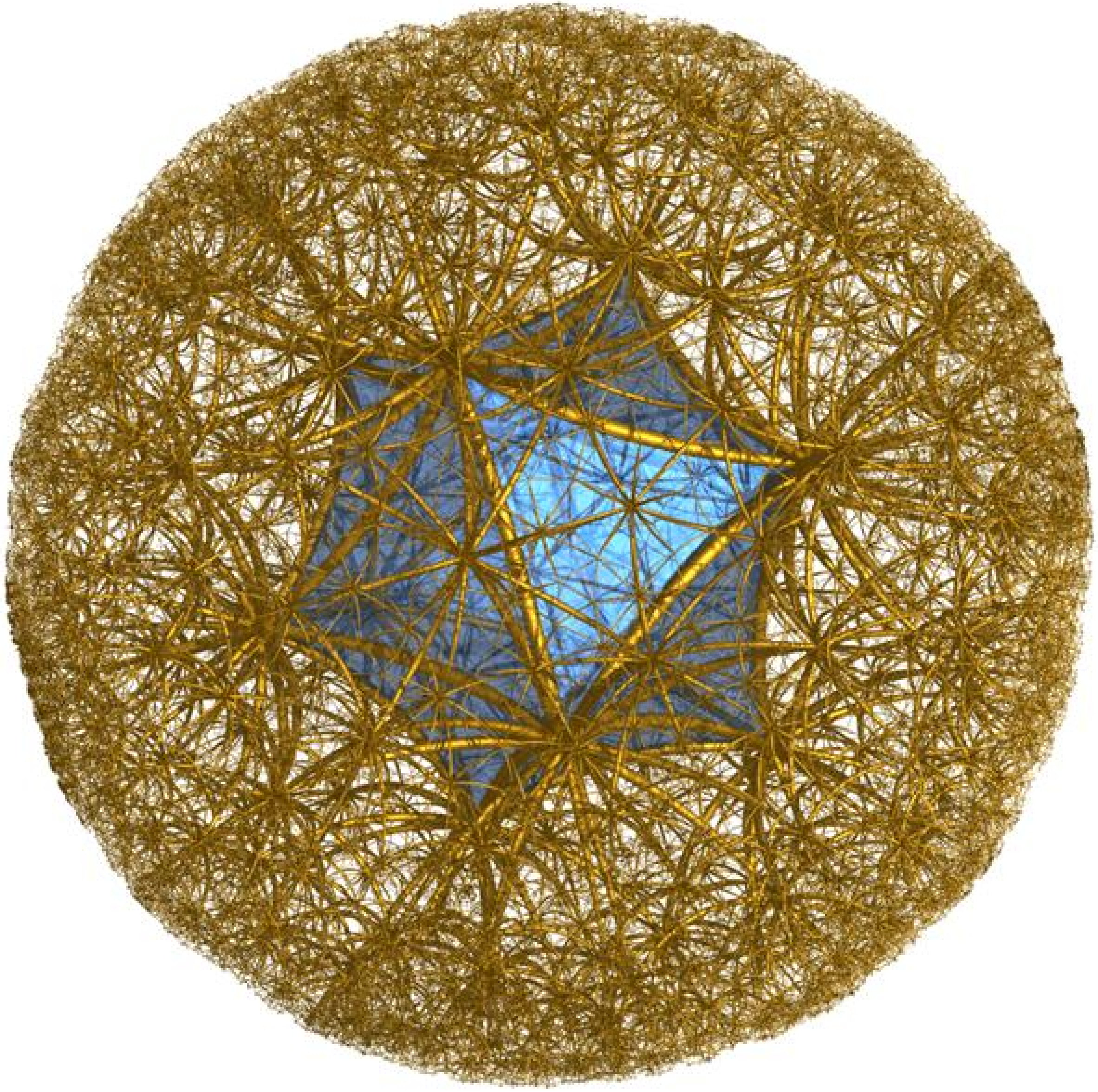
Visualization of tiling in H^3



Just kidding!

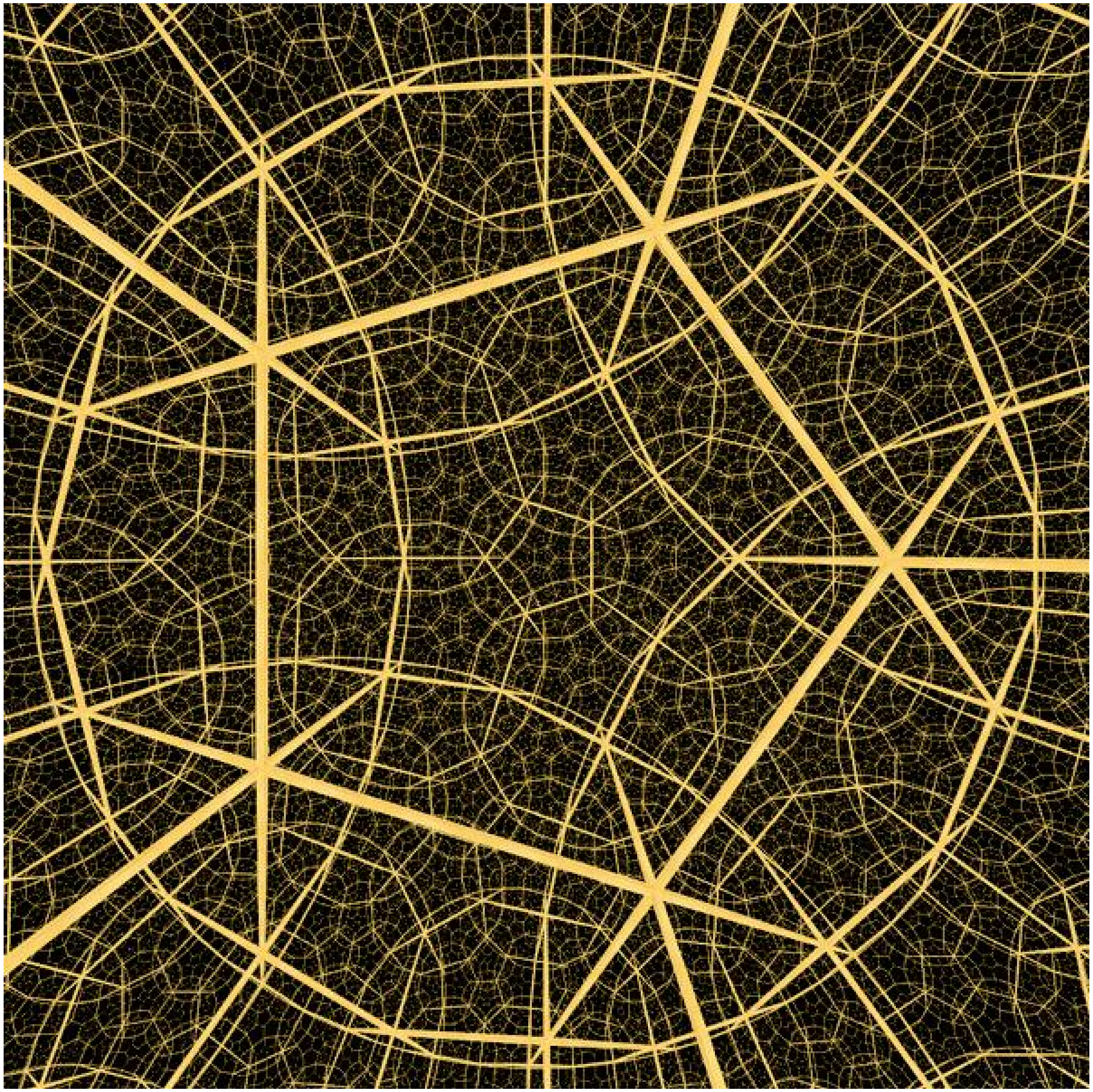
It is a sphere, which is good approximation to 10^{10} dodecahedral tiles.

Visualization of tiling in H^3



Different approach - rendering only edges
(image by Claudio Rocchini from wikipedia.org).
4 generations of tiles are shown. Not much is possible to see.

Visualization of tiling in H^3

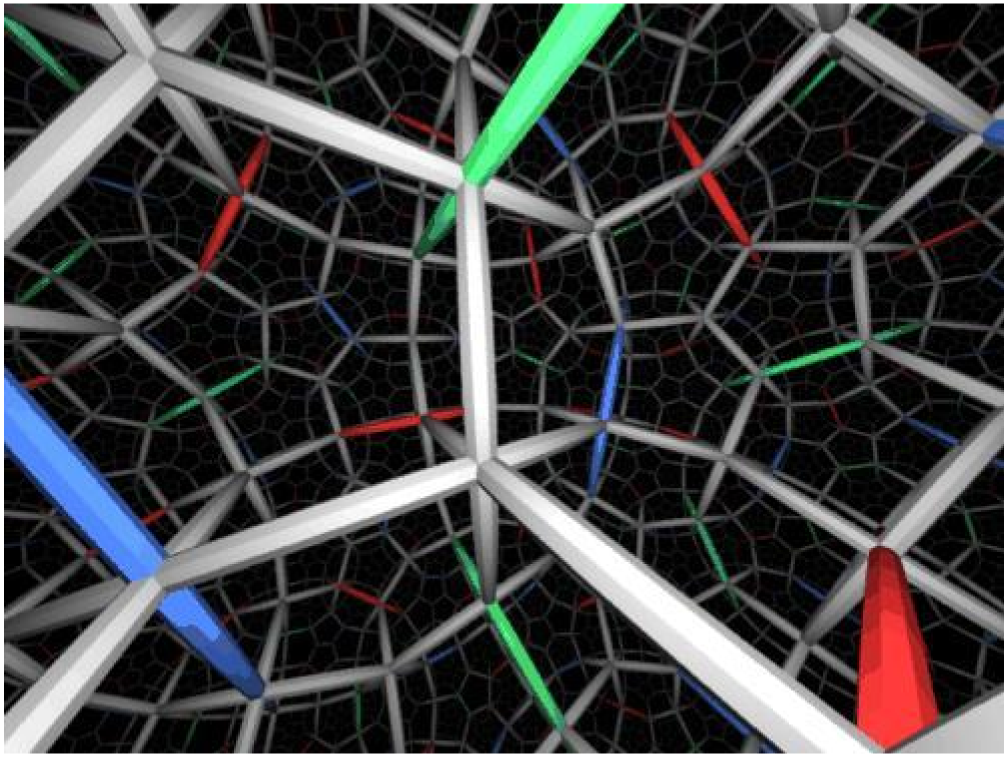


Another approach - looking at the tiling from inside of the space (W. Thurston, J. Weeks). This allows to see clearly the local structure of the tiling.

Image by Tom Ruen from wikipedia.org generated by Jeff Weeks interactive software.

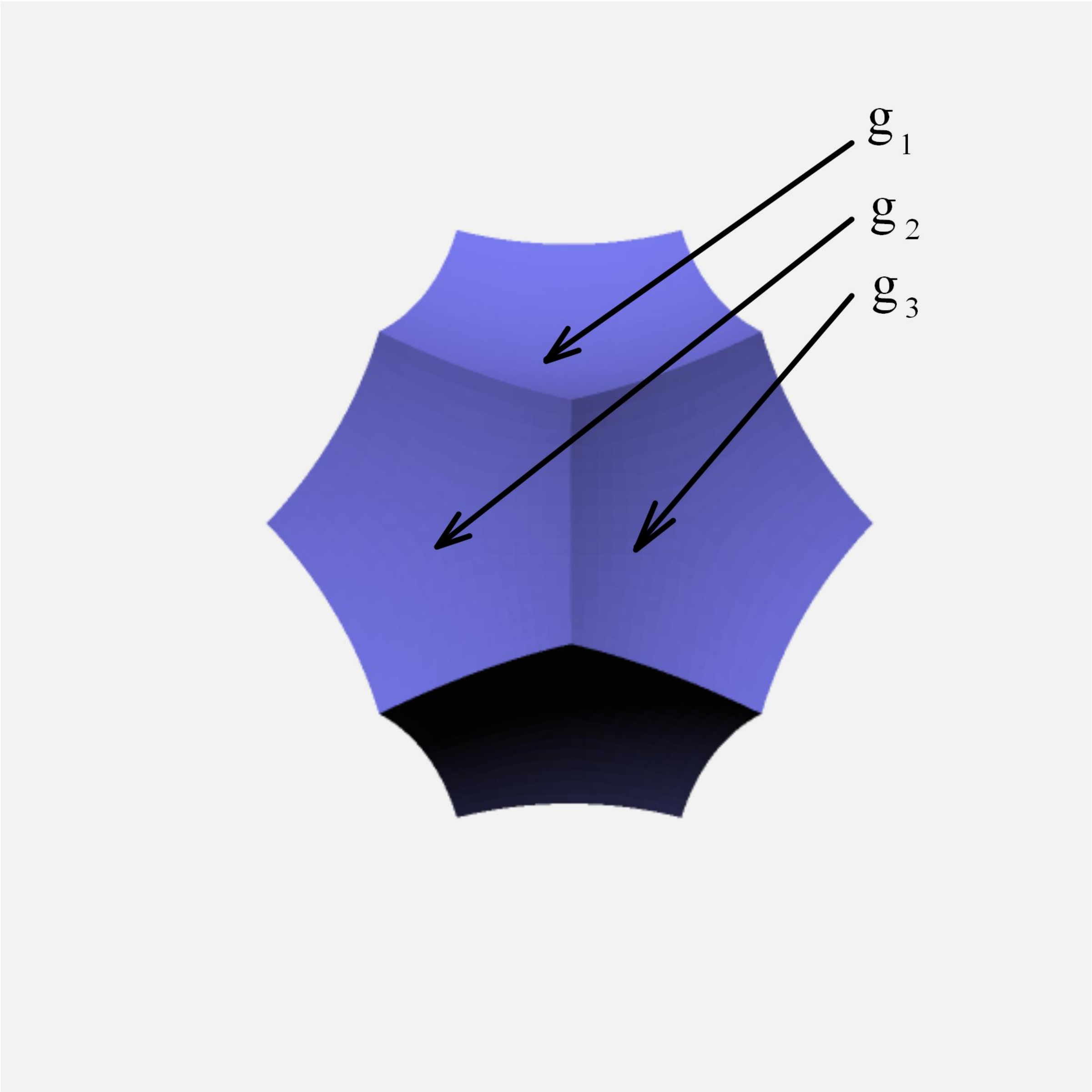
Similar image from "Knot plot" video was used on cover of dozens of math books.

Visualization of 3D hyperbolic tilings



Hyperbolic space tessellation as it looks from inside the space. Frame from "Not Knot" video (1994)

Visualization of tiling by subgroup



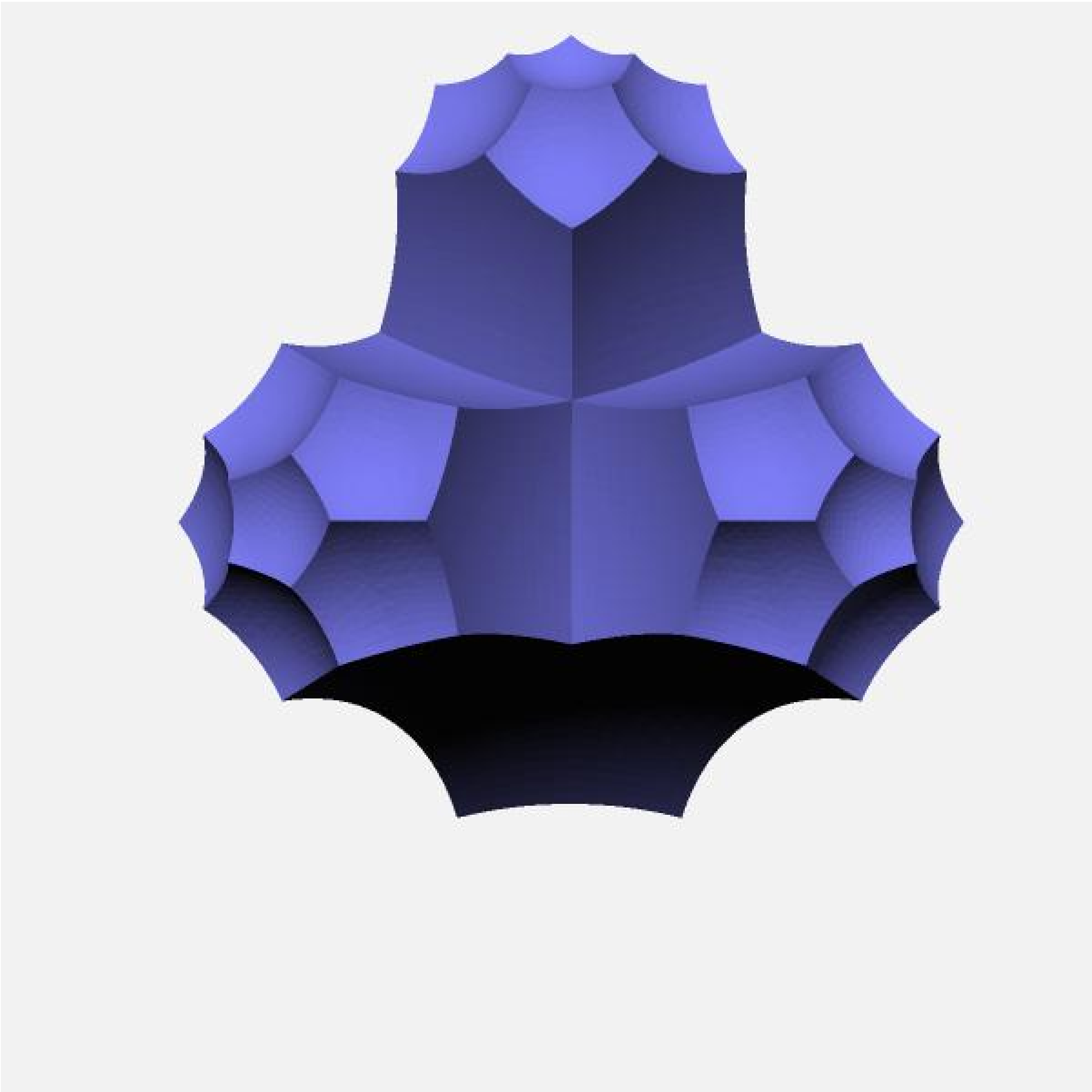
Looking from inside we can see only nearest neighbours.

Looking from outside we can only see the outer boundary.

Let's try to look outside, but see not the whole tiling, but only it's part.

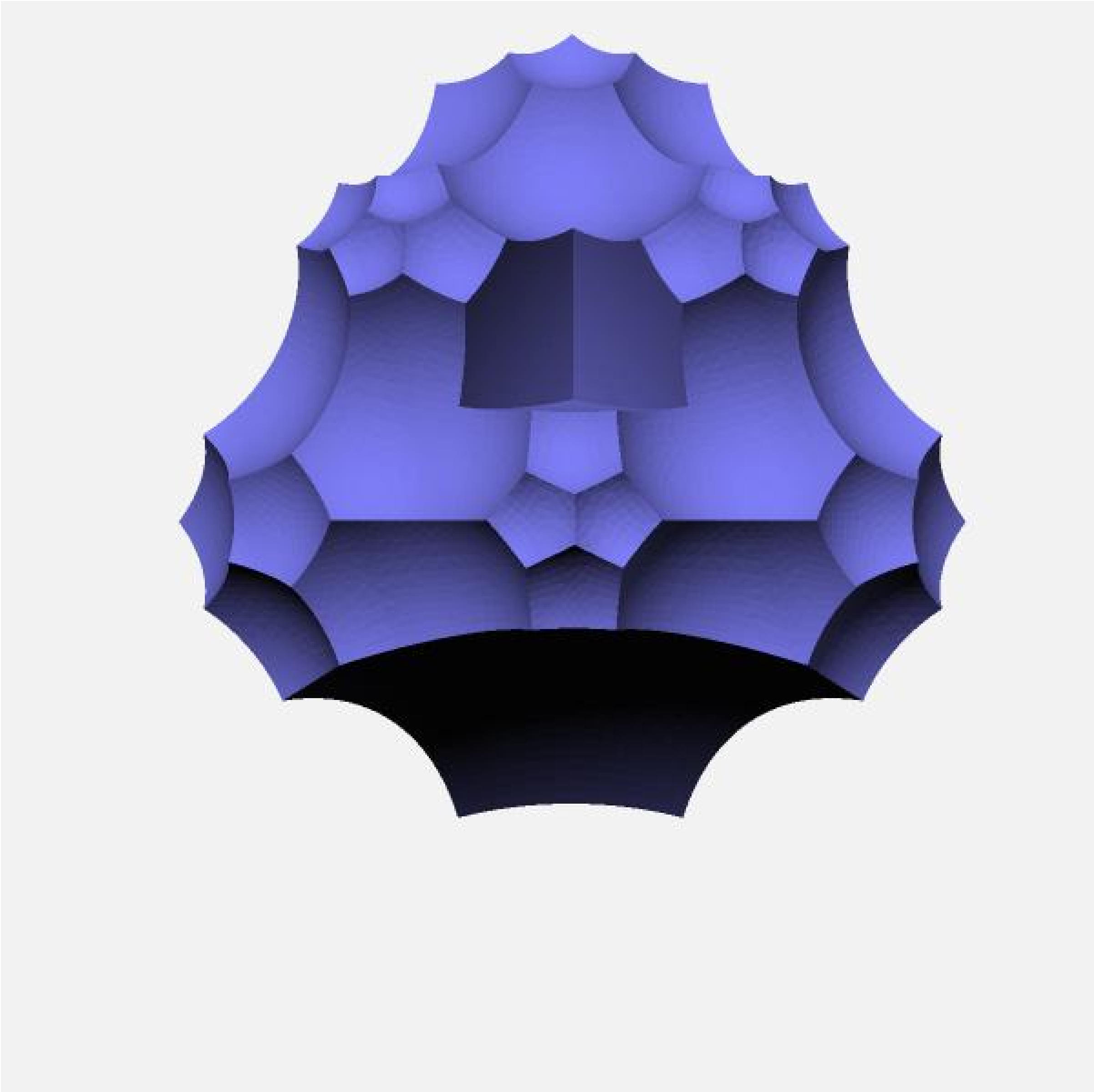
We select only subset of all generators. The subset will generate subgroup of the whole group.

Visualization of tiling by subgroup



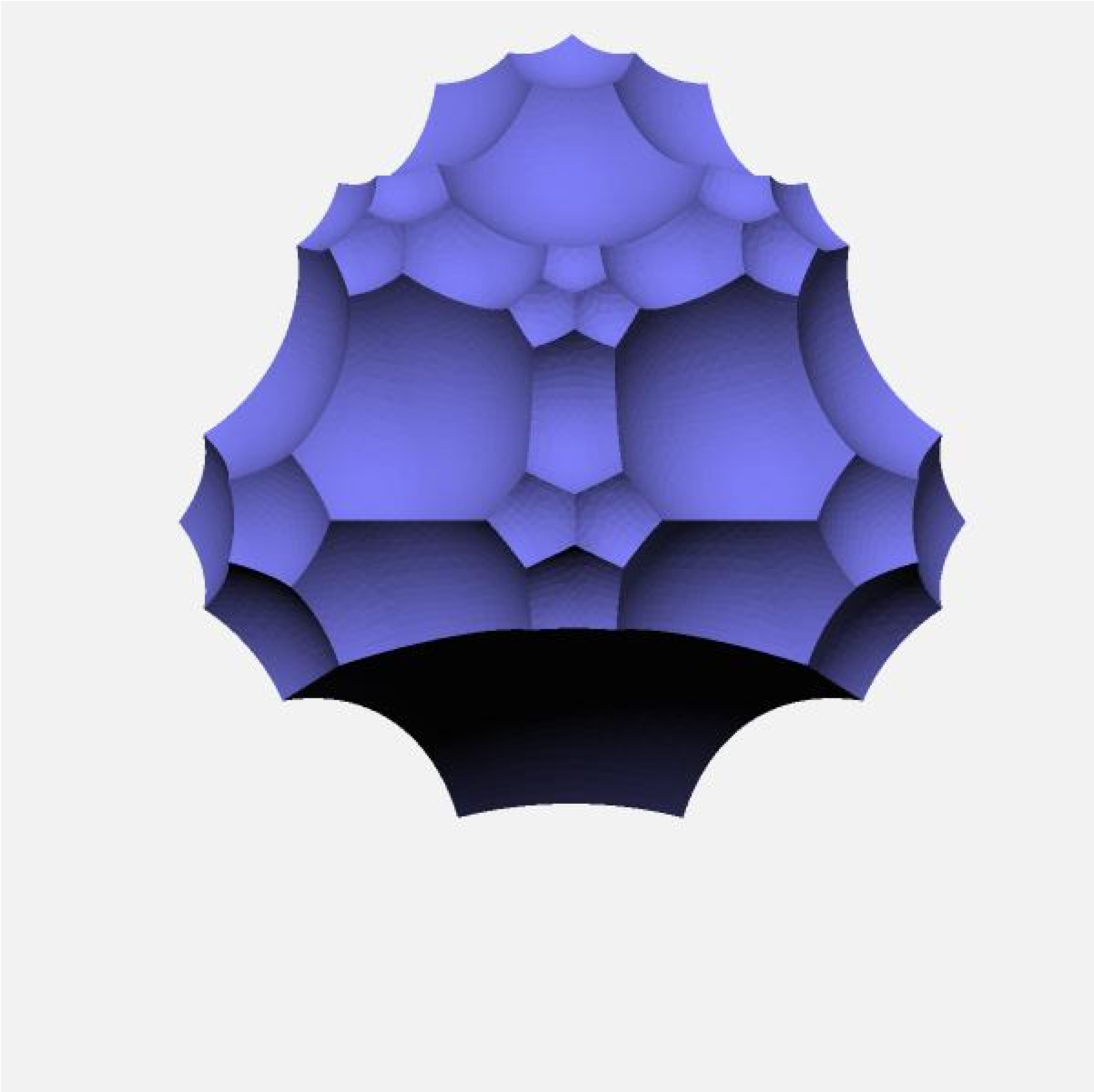
First iteration of subgroup generation.

Visualization of tiling by subgroup



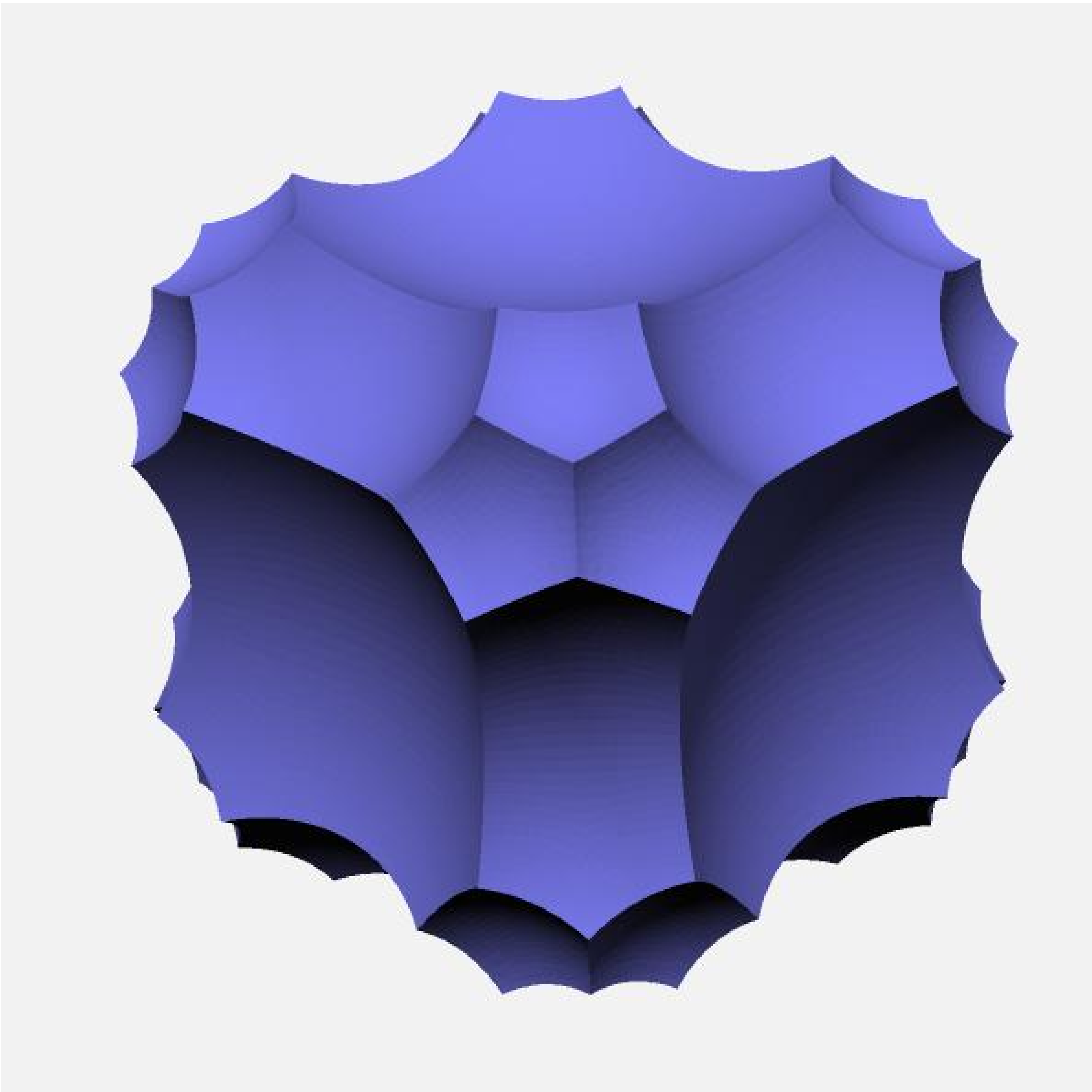
Second iteration of subgroup generation.

Visualization of tiling by subgroup



Third (and last) iteration of subgroup generation.

Visualization of tiling by subgroup

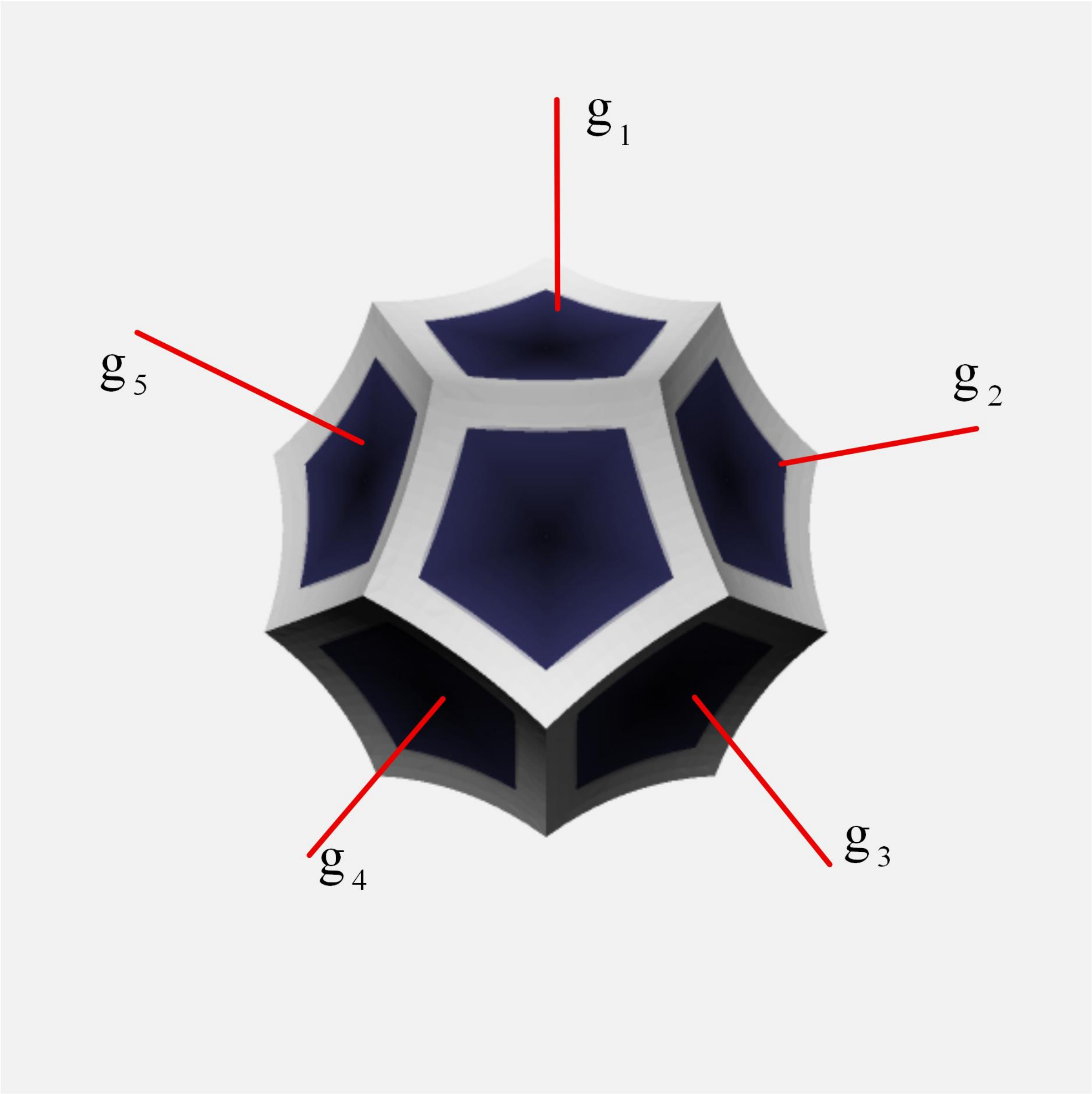


Move center of the shape into the center of the Poincare ball. Shape has obvious cubic symmetry.

Visualization of tiling by subgroup

Another view of the same finite subtiling. We have 8 dodecahedra around center

Visualization of infinite subgroup

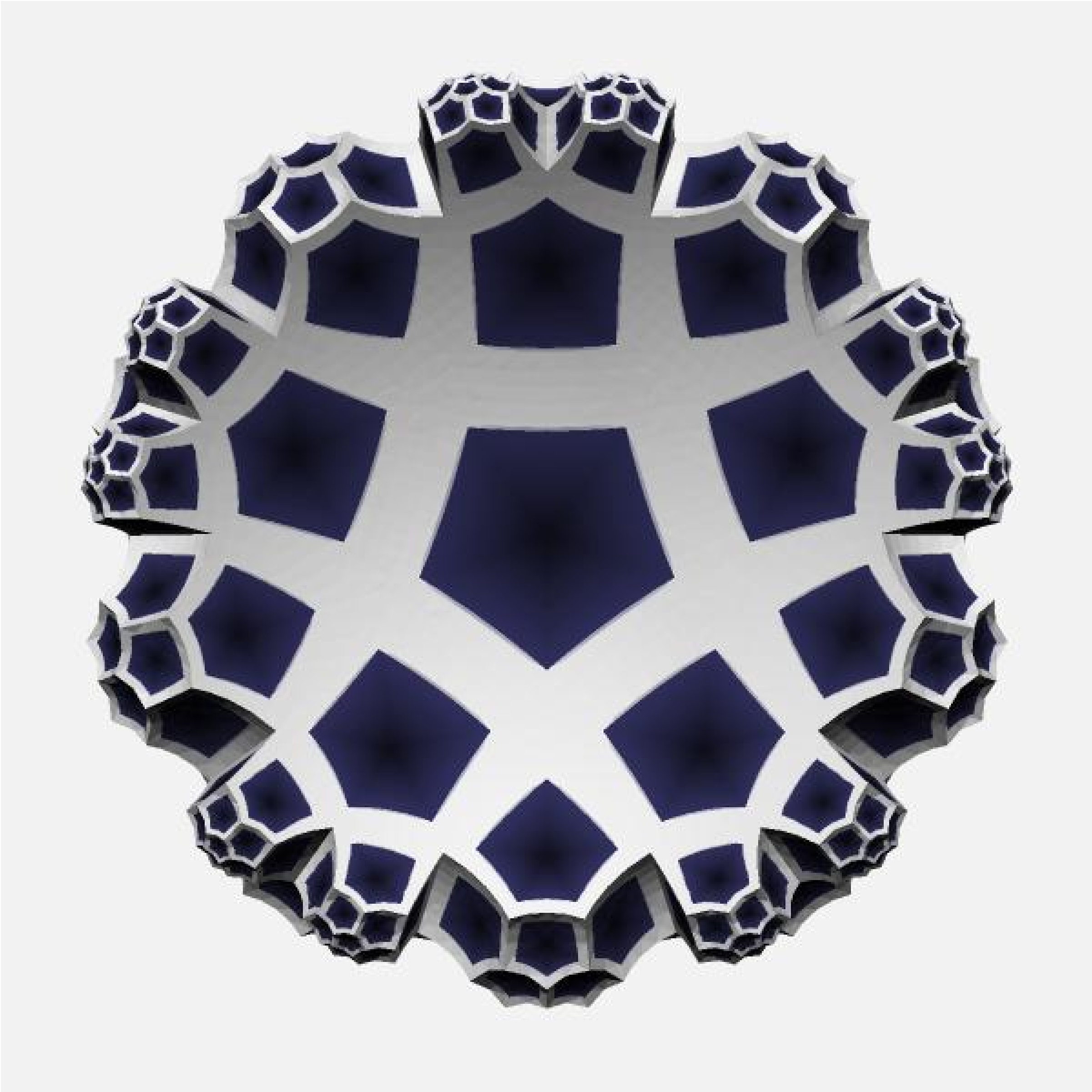


We can select generators, which generate infinite subgroup.

Visualization of infinite subgroup

First iteration.

Visualization of infinite subgroup



Second iteration.

Visualization of infinite subgroup

Third iteration.

Infinite Fuchsian subgroup

Few more iterations.

All tiles are lying in one hyperbolic plane.

Infinite Fuchsian subgroup

Rotation of the tiling around the center of the ball.

Infinite Fuchsian subgroup

Rotation of the tiling around the center of the ball.

Infinite Fuchsian subgroup

Rotation of the tiling around the center of the ball.

Infinite Fuchsian subgroup

Rotation of the tiling around the center of the ball.

Infinite Fuchsian subgroup

Hyperbolic translation of plane into the center of the ball.

Infinite Fuchsian subgroup

Hyperbolic translation of plane into the center of the ball.

Infinite Fuchsian subgroup

More rotation about the center of the ball.

Infinite Fuchsian subgroup

More rotation about the center of the ball.

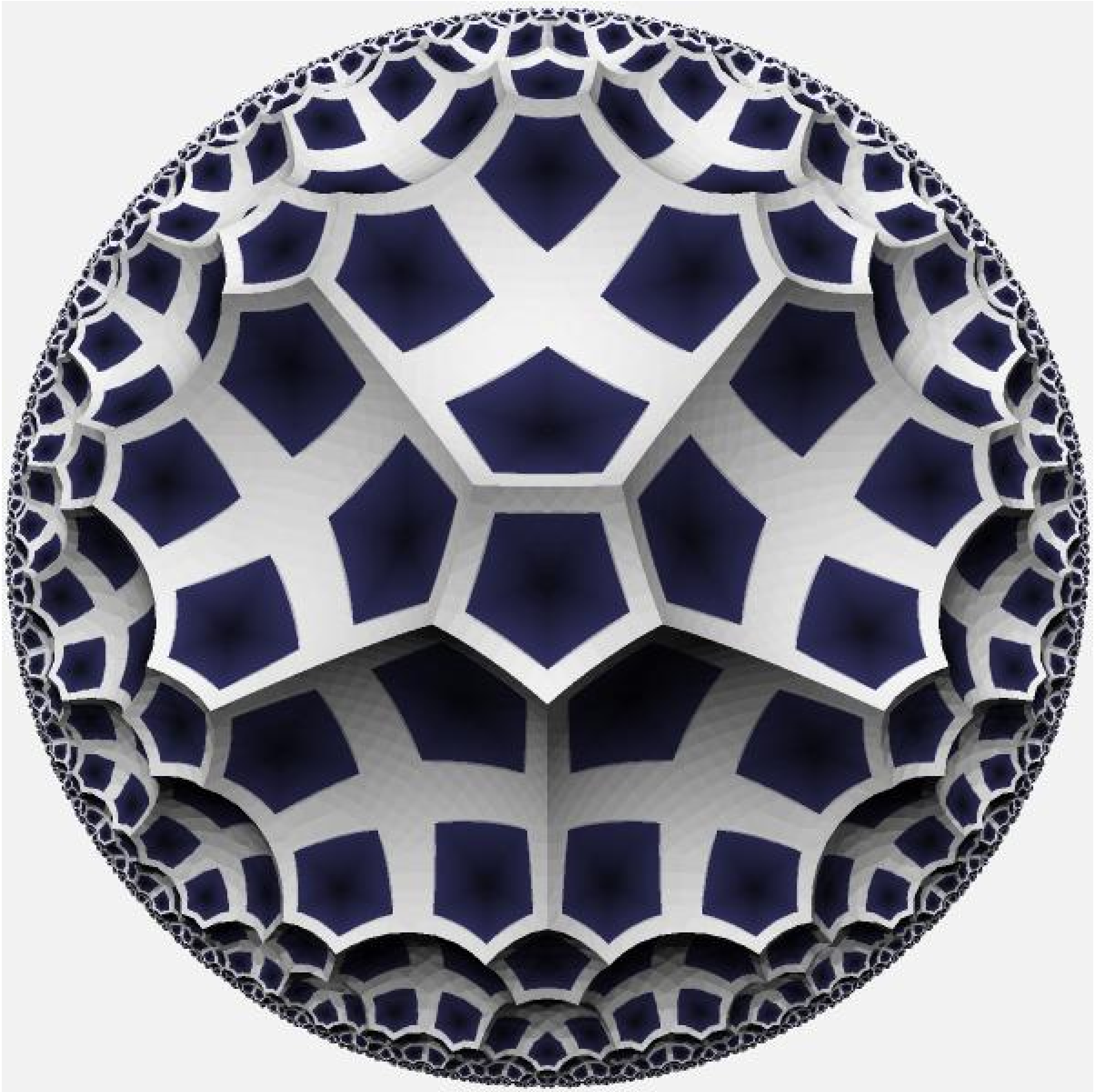
Infinite Fuchsian subgroup

More rotation about the center of the ball.

Infinite Fuchsian subgroup

More rotation about the center of the ball.

Infinite Fuchsian subgroup



More rotation about the center of the ball.

Tiling in Cylinder Model

Stretching the Poincare ball into infinite cylinder.

Tiling in Cylinder Model

Stretching the Poincare ball into infinite cylinder.

Tiling in Cylinder Model

Stretching the Poincare ball into infinite cylinder.

Tiling in Cylinder Model

Stretching the Poincare ball into infinite cylinder.

Tiling in Cylinder Model

Stretching the Poincare ball into infinite cylinder.

Tiling in Cylinder Model

Stretching the Poincare ball into infinite cylinder.

Tiling in Cylinder Model

Cylinder model is straighforward axially symmetric 3D generalization of the conformal band model of the hyperbolic plane.

Cylinder model of hyperbolic space is not conformal, it has some limited angular distorsions. However, it has such a nice property as euclidean metrics along cylinder's axis. It is especially usefull when we want to visualize some specific hyperbolic geodesic which we aling in that case with the cylinder's axis. All the planes orthoginal to that geodesic are represented by circular disks orthogonal to the axis.

Tiling in Cylinder Model

Different orientation of the same tiling inside of cylinder model of hyperbolic space.

If cylinder's axis is aligned with the axis of a hyperbolic transformation of the group, tiling will have euclidean translational symmetry along the cylinder's axis. If it is aligned with the axis of a loxodromic transformation, tiling will spiral around the cylinder's axis.

Tiling in Cylinder Model

Different orientation of the same tiling inside of cylinder model of hyperbolic space.

This group has no loxodromic transformatons.

Tiling in Cylinder Model

Different orientation of the same tiling inside of cylinder model of hyperbolic space.

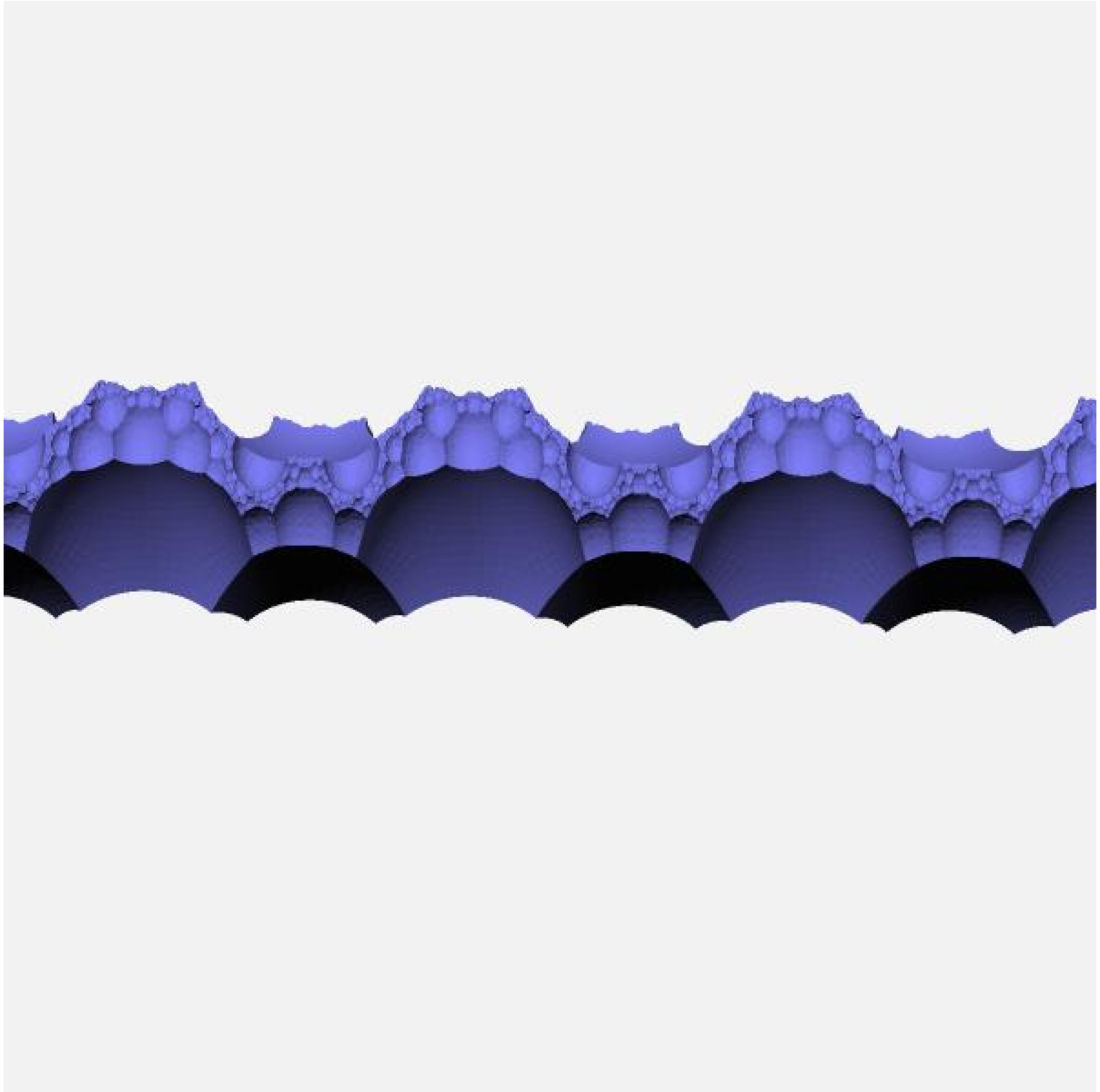
Transformation with longer period is aligned with the cylinder's axis.

Tiling in Cylinder Model

Different orientation of the same tiling inside of cylinder model of hyperbolic space.

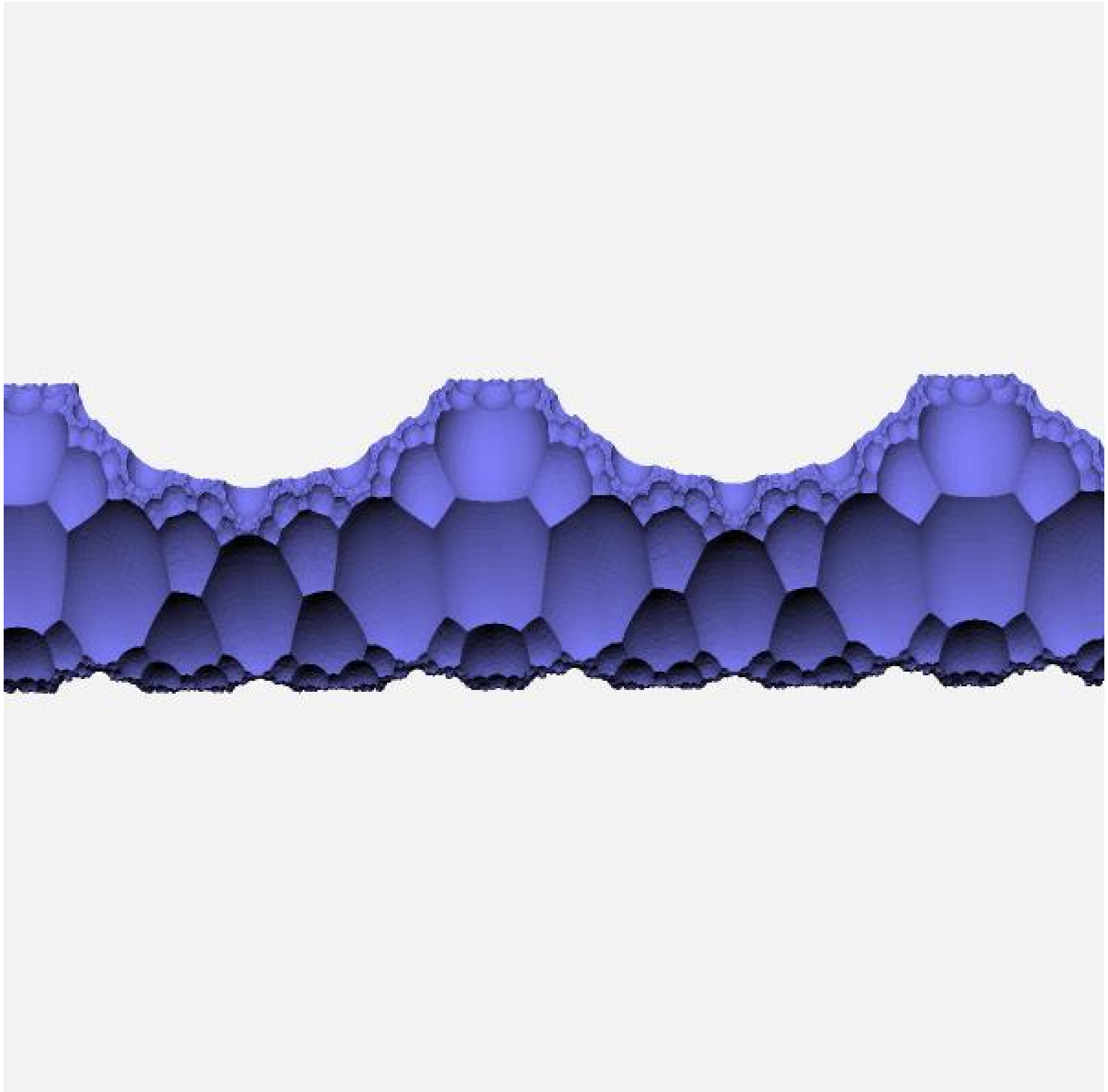
No period is visible.

Quasifuchsian Subgroup



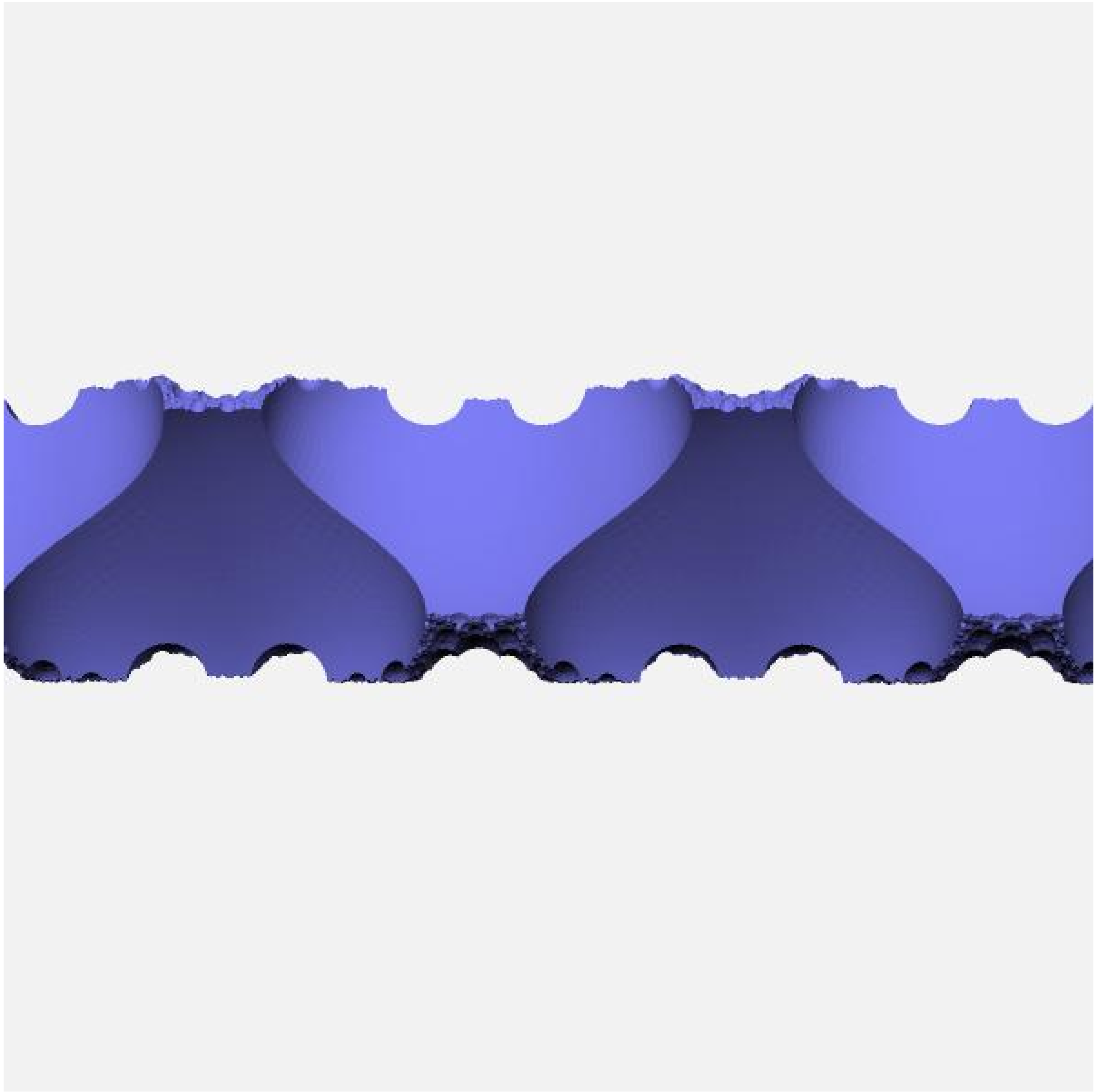
Different subset of generators generates tiling, which is not flat anymore.
The limit set of the tiling is continous fractal curve. Axis of cylinder is aligned with the axis of an hyperbolic transformation of the subgroup.

Quasifuchsian Subgroup



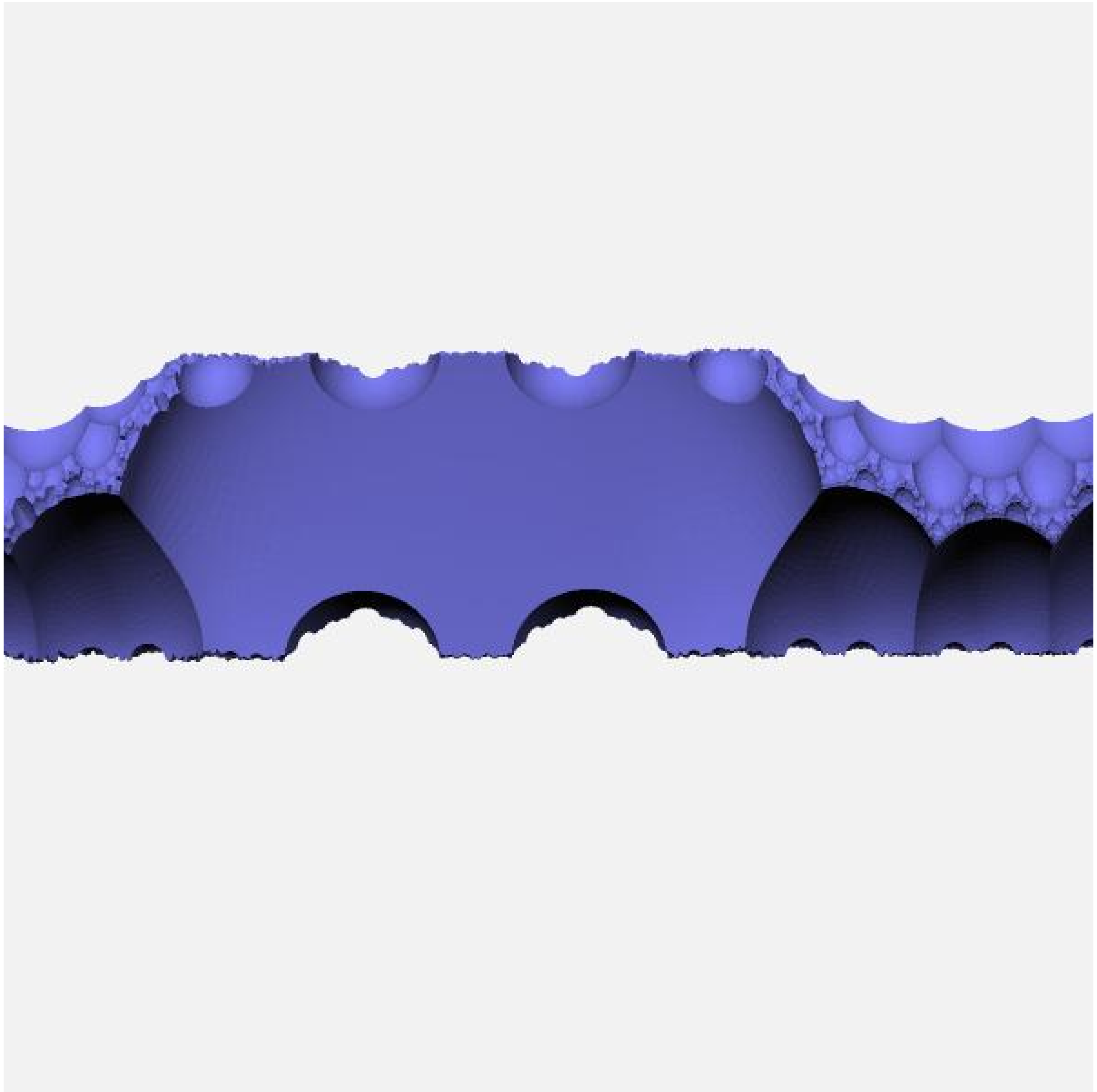
Another orientation of the same tiling.
Axis of cylinder is aligned with the axis of an hyperbolic transformation of the subgroup.
Tiling is periodic along cylinder's boundary.

Quasifuchsian Subgroup



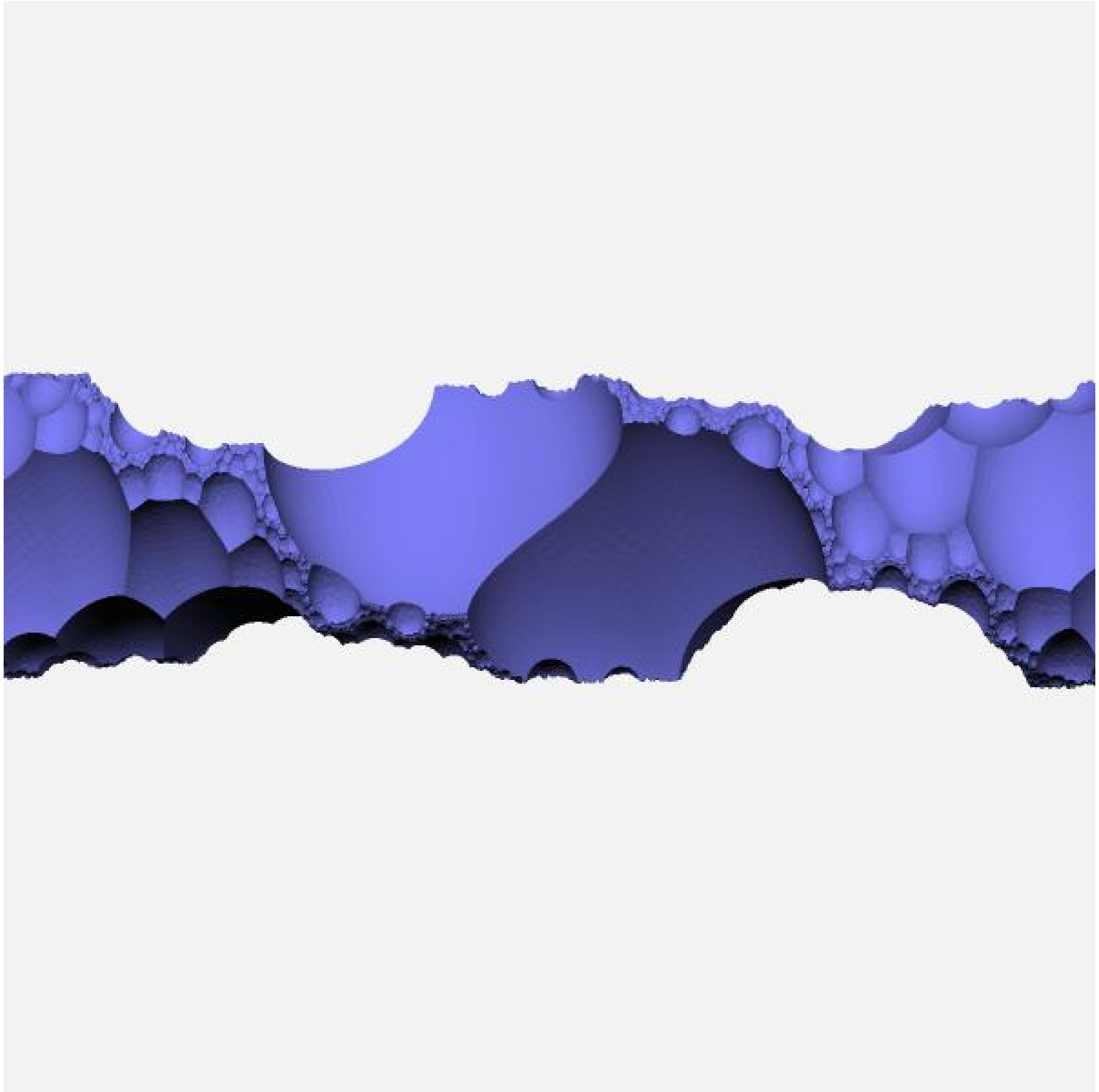
Another orientation of the same tiling.
Axis of cylinder is aligned with the axis of an hyperbolic transformation of the subgroup.
Tiling is periodic along cylinder's boundary.

Quasifuchsian Subgroup



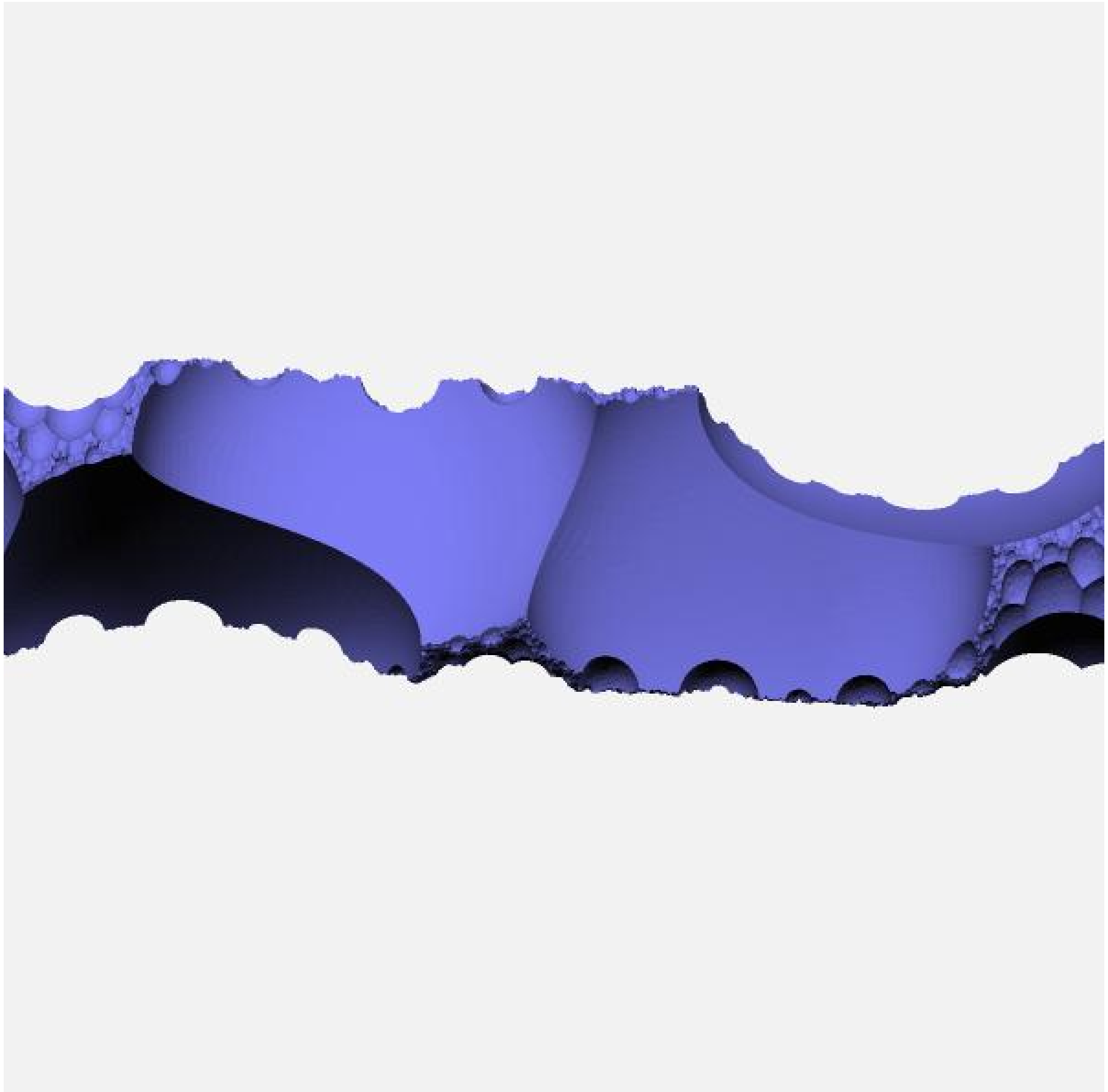
Another orientation of the same tiling.
Axis of cylinder is aligned with the axis of an hyperbolic transformation of the subgroup.
Tiling is periodic along cylinder's boundary.

Quasifuchsian Subgroup



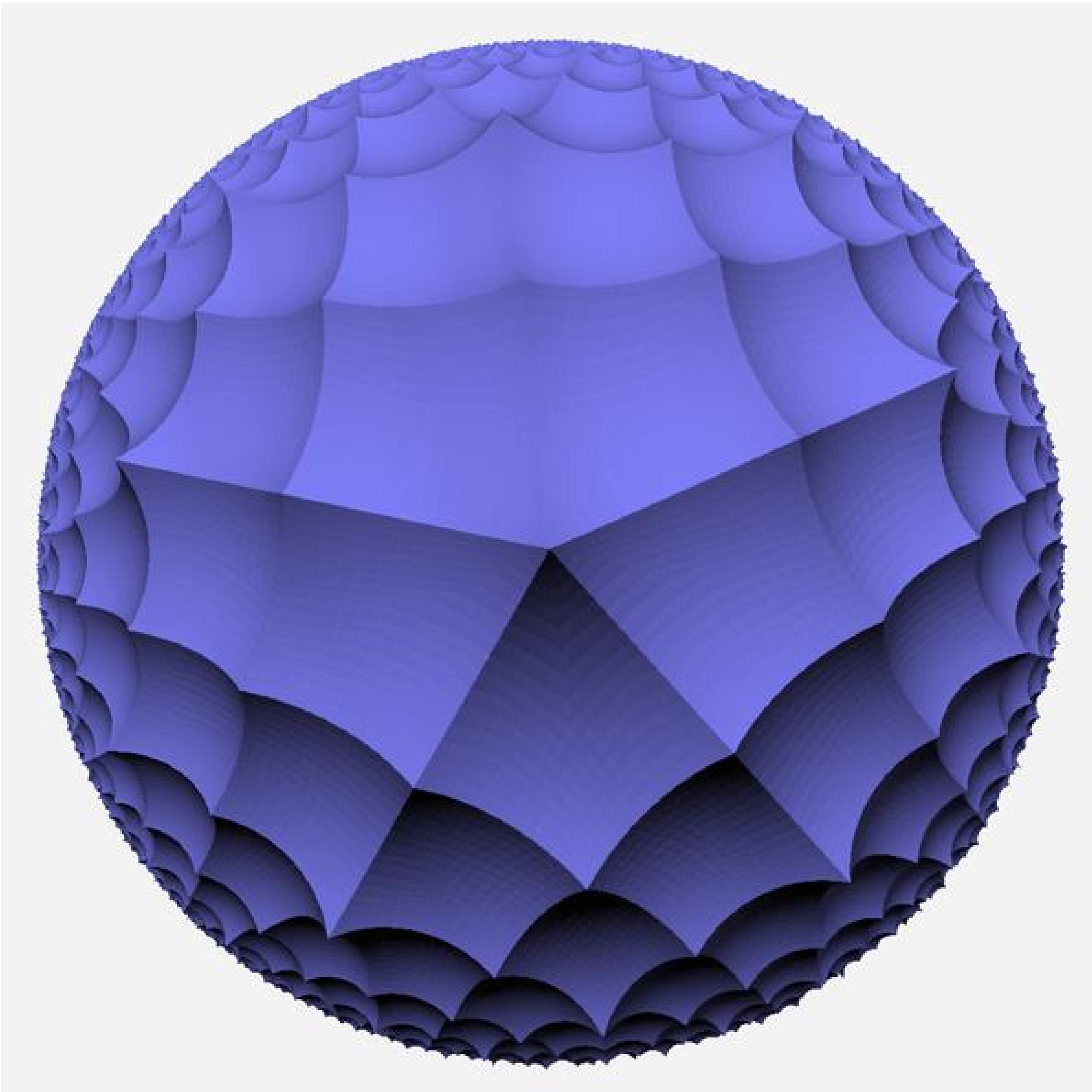
Axis of cylinder is aligned with the axis of a loxodromic transform. Tiling is spiraling around the cylinder's axis.

Quasifuchsian Subgroup



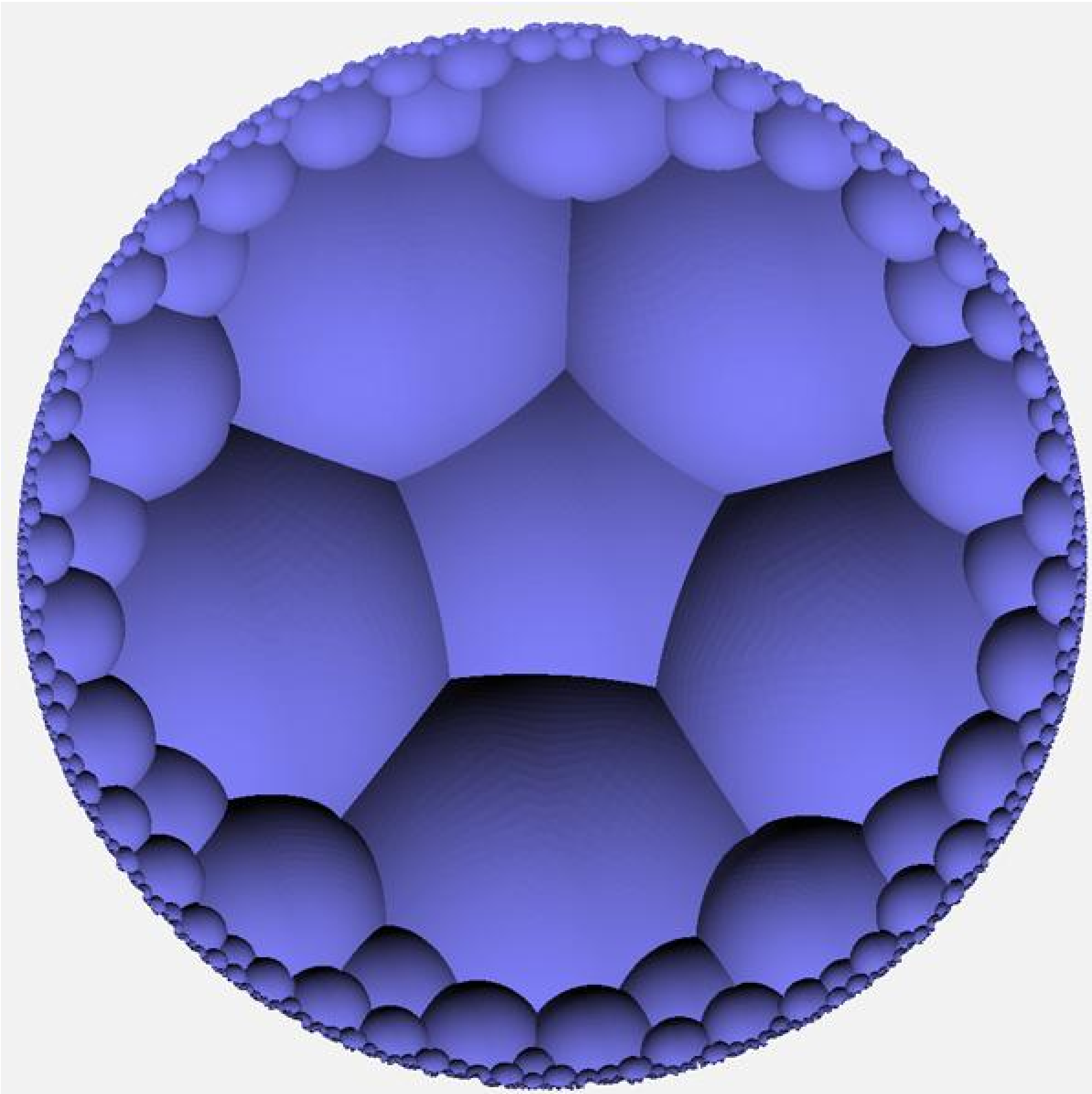
Axis of cylinder is aligned with the axis of different loxodromic transform.
Tiling is spiraling around the cylinder's axis at different speed.

Samples of 3D hyperbolic tilings



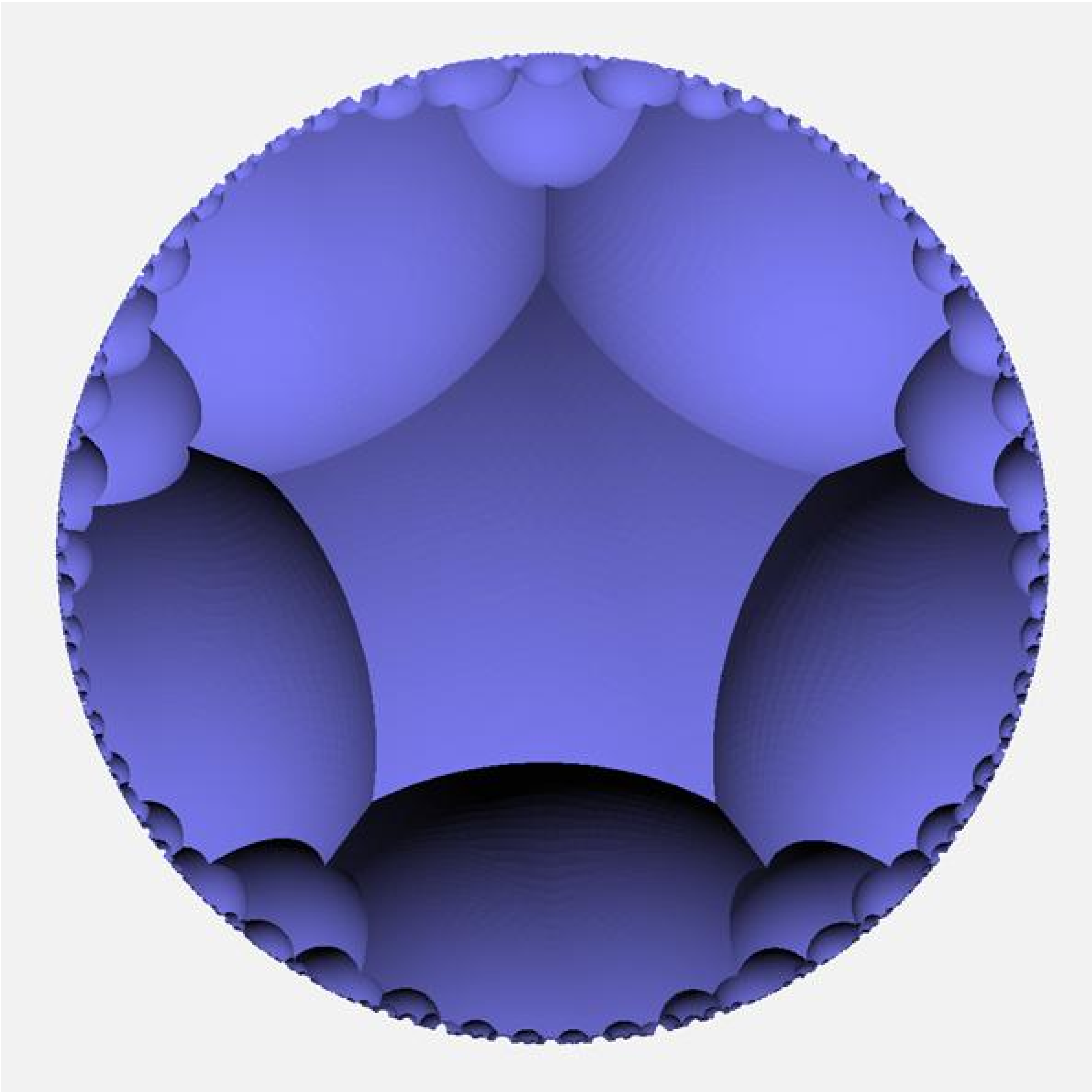
Quasifuchsian tiling with Lambert cubes show in the Poincare ball model. Only 4 out of 6 generators are used.

Samples of 3D hyperbolic tilings



Quasifuchsian tiling with 5 and 9 sided prisms.

Samples of 3D hyperbolic tilings

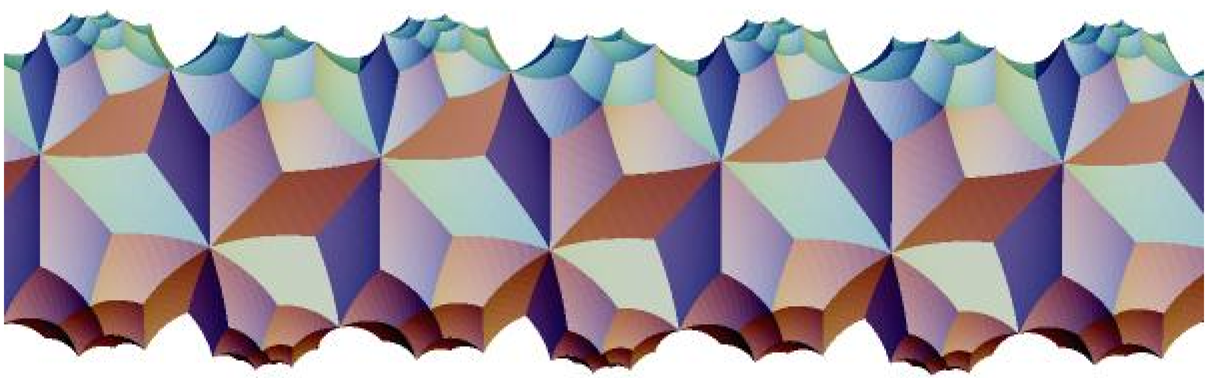


Quasifuchsian tiling with 5 and 18 sided prisms.

Samples of 3D hyperbolic tilings

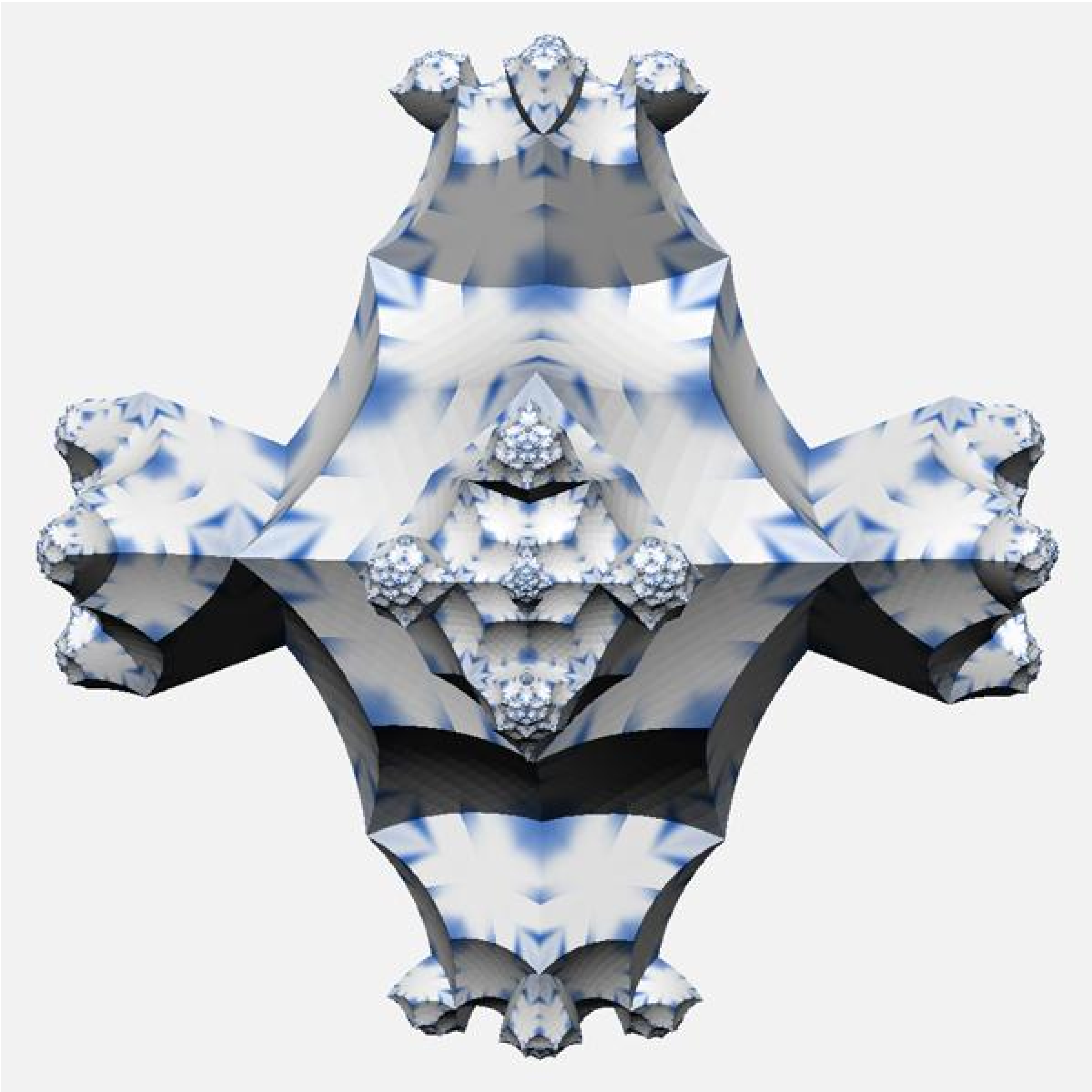
Tiling with right angled dodecahedra.
Limit set is point set.

Samples of 3D hyperbolic tilings



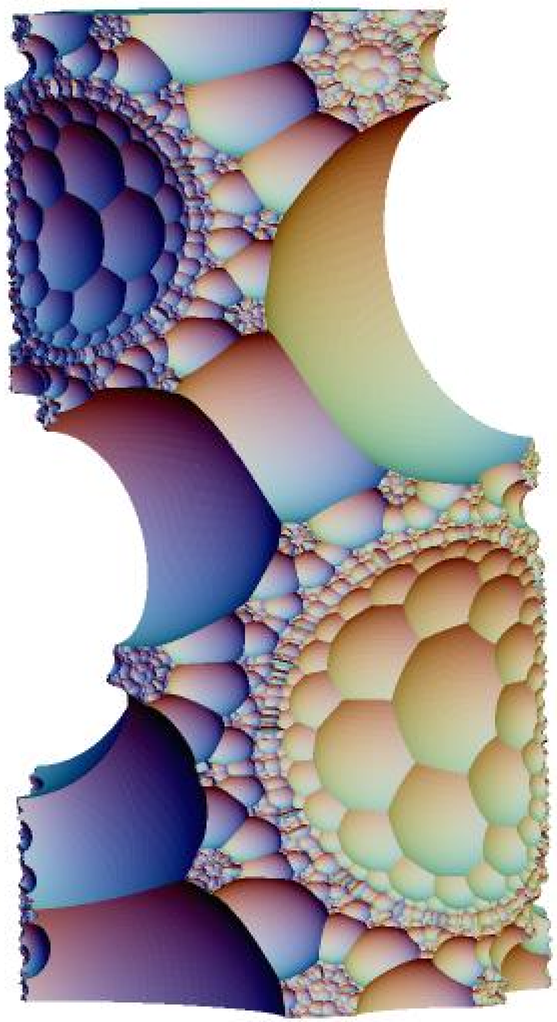
Tiling with rhombic triacontahedra in the cylinder model.

Samples of 3D hyperbolic tilings



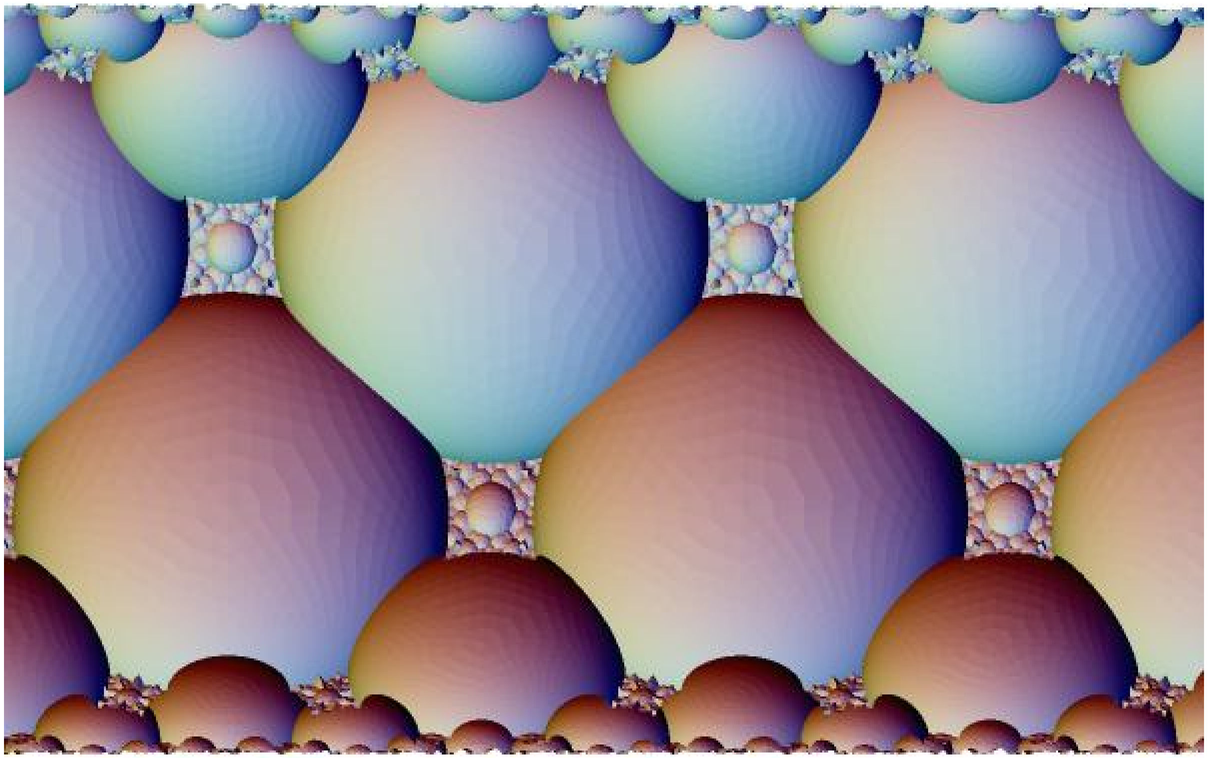
Tiling with rhombic triacontahedra

Samples of 3D hyperbolic tilings



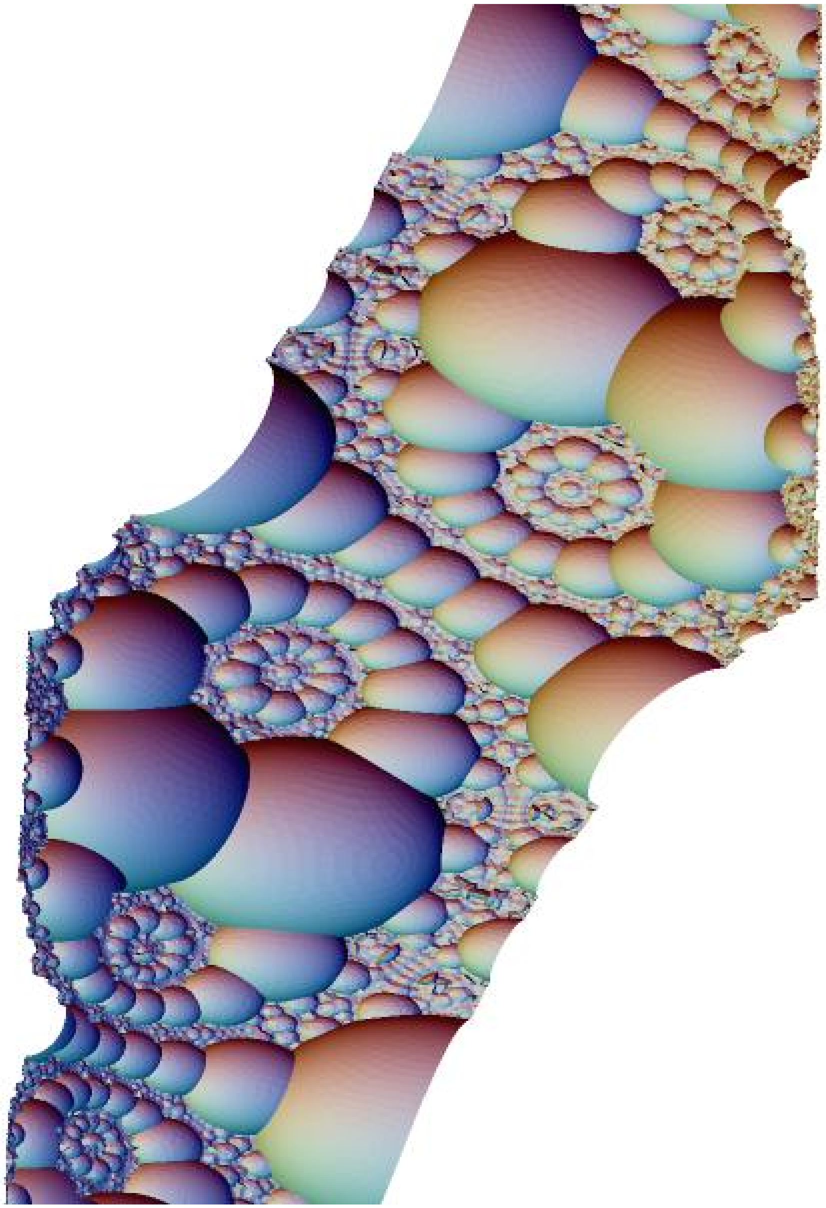
Tiling with truncated tetrahedra. Axis of cylinder model is vertical and it is aligned with common perpendicular to two selected planes. These planes become flat in the cylinder model and are at the top and bottom of the piece.

Samples of 3D hyperbolic tilings



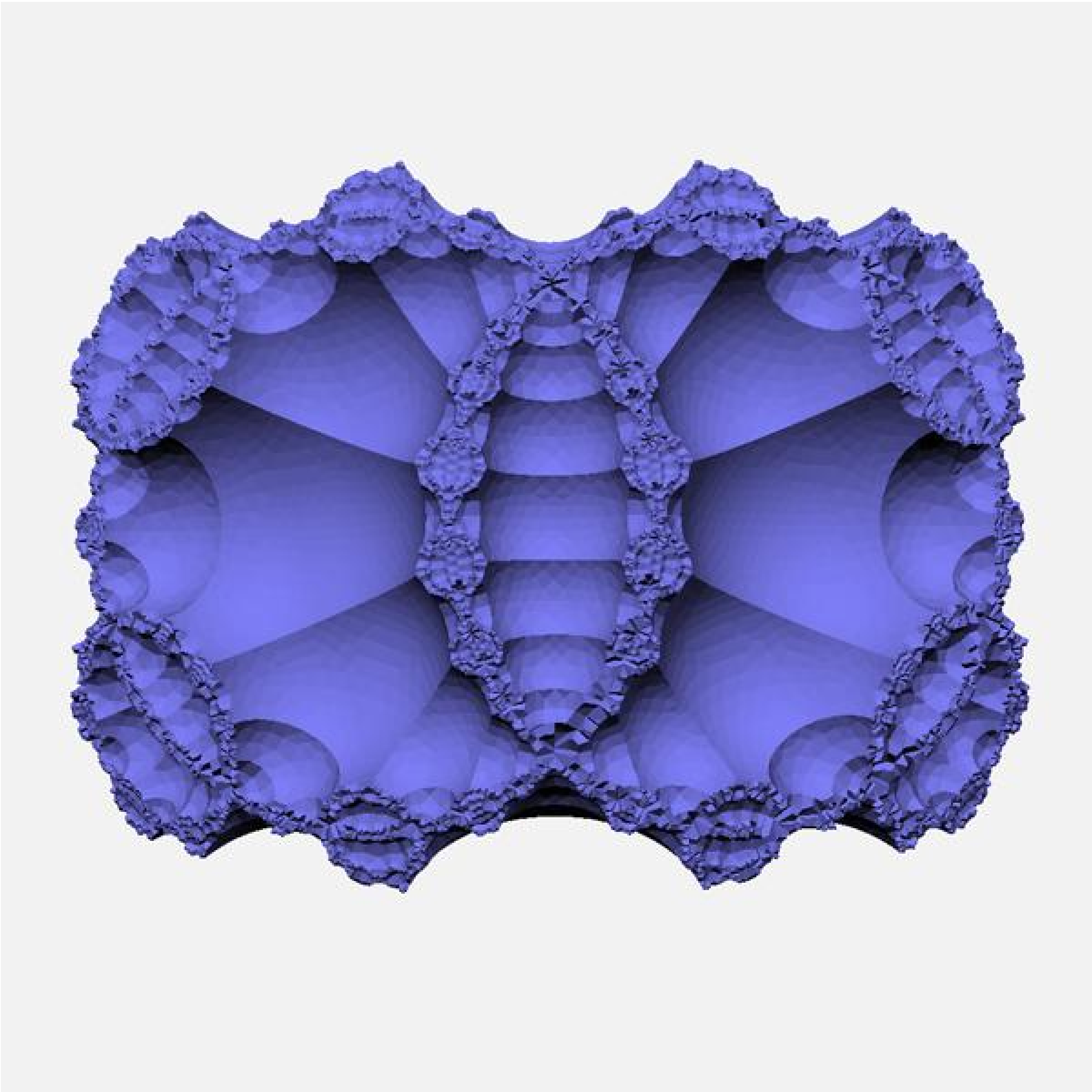
Tiling with different truncated tetrahedra in cylinder model. Axis of cylinder is aligned with axis of some hyperbolic transformation of the group, which makes tiling periodic.

Samples of 3D hyperbolic tilings



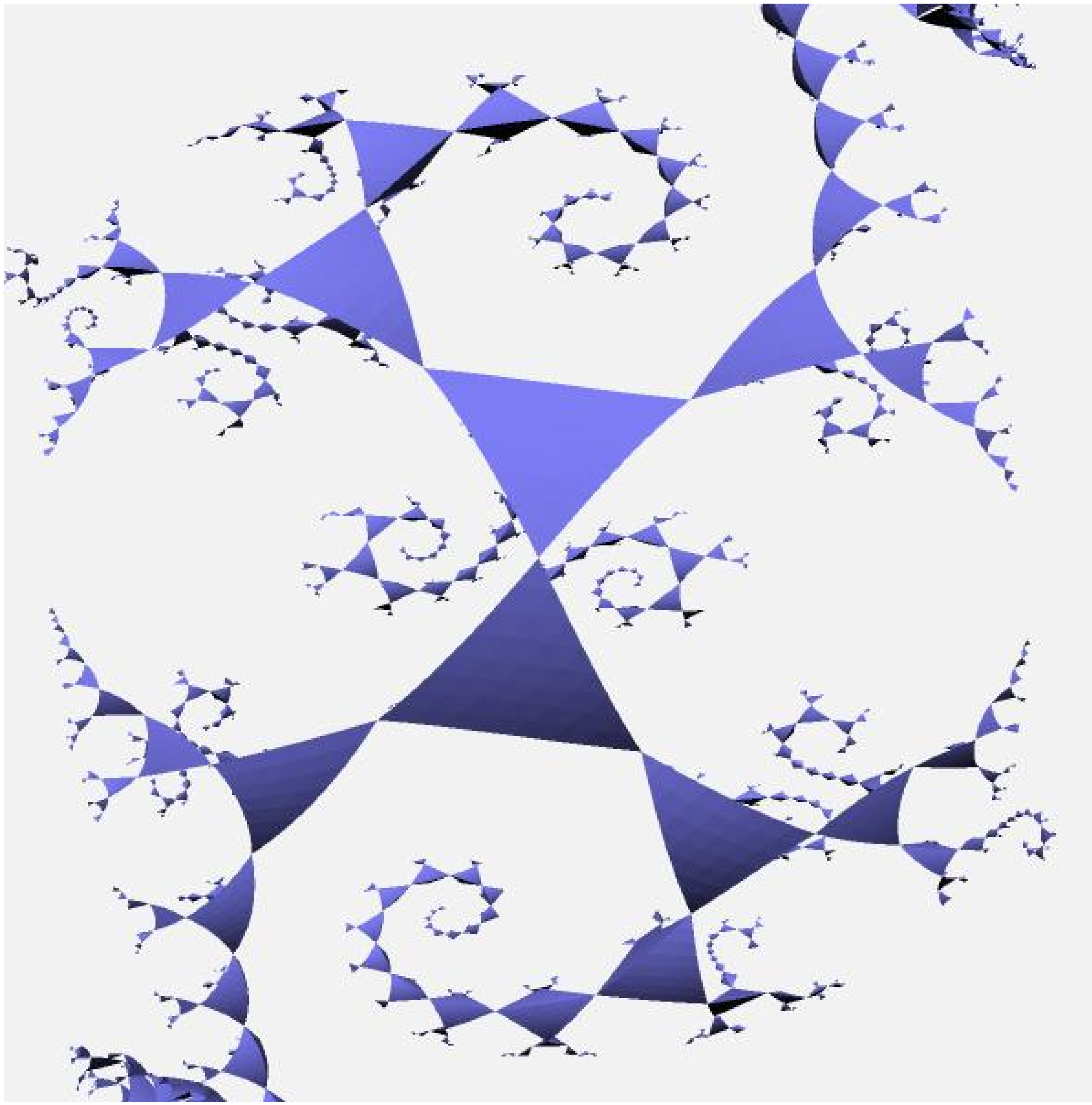
Tiling with truncated Lambert cubes.
The limit set is complex fractal.

Samples of 3D hyperbolic tilings



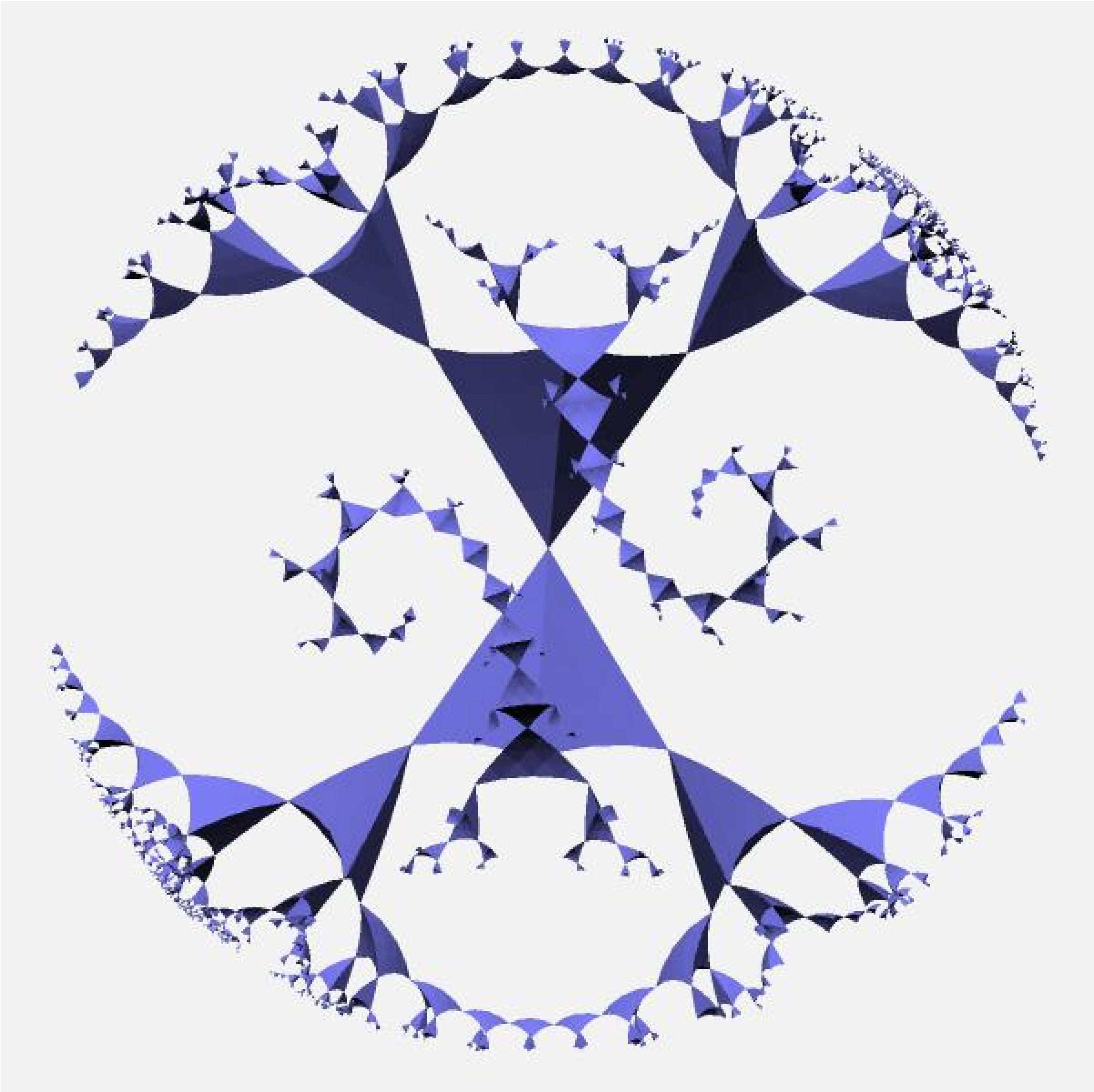
Another tiling with truncated cubes.

Samples of 3D hyperbolic tilings



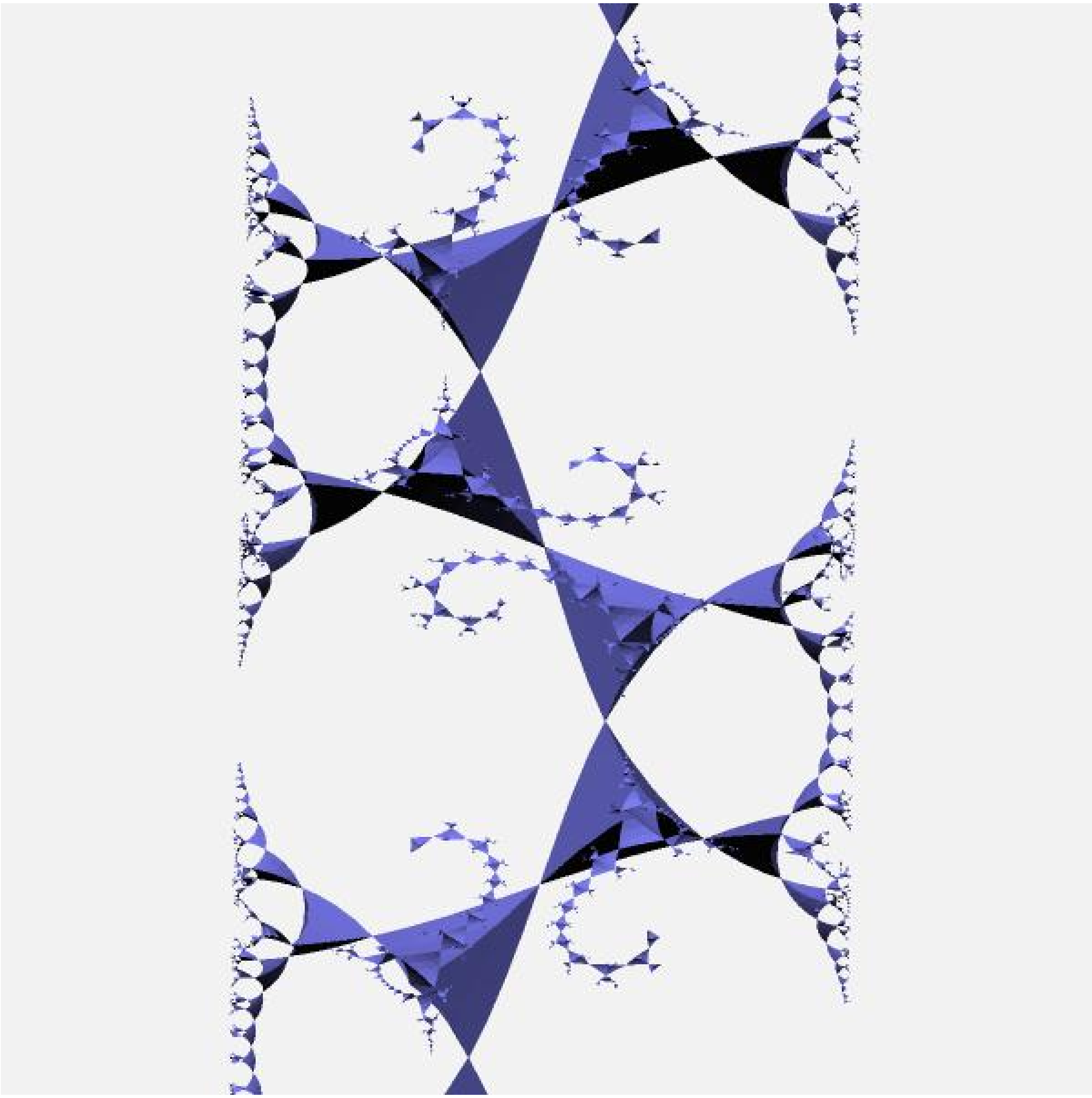
Tiling with two generator free group.
Shown in the cylinder model.
Generators have parabollic commutator,
which is responsible for cusps.

Samples of 3D hyperbolic tilings



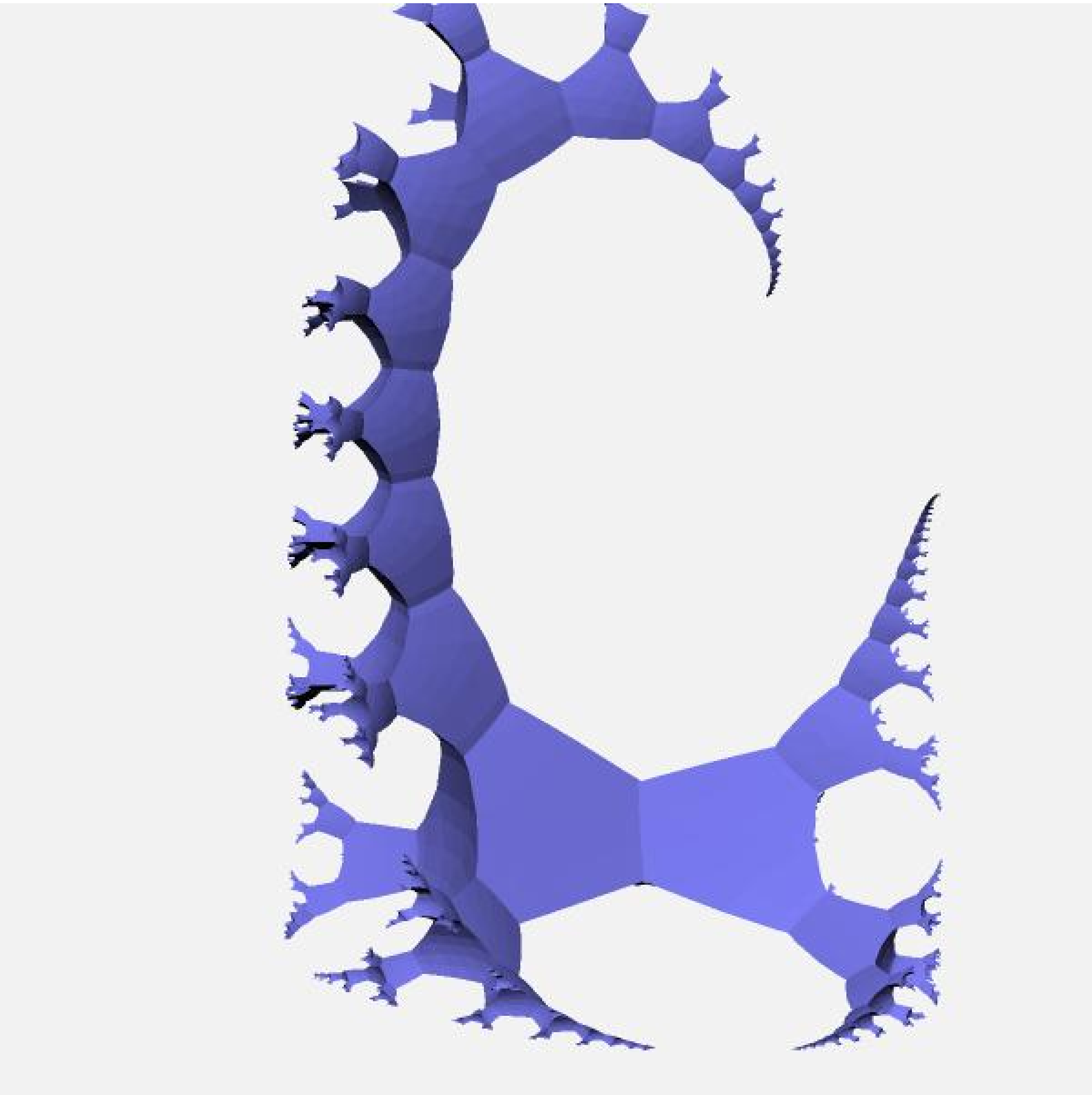
The same tiling in shown in the Poincare ball model.

Samples of 3D hyperbolic tilings



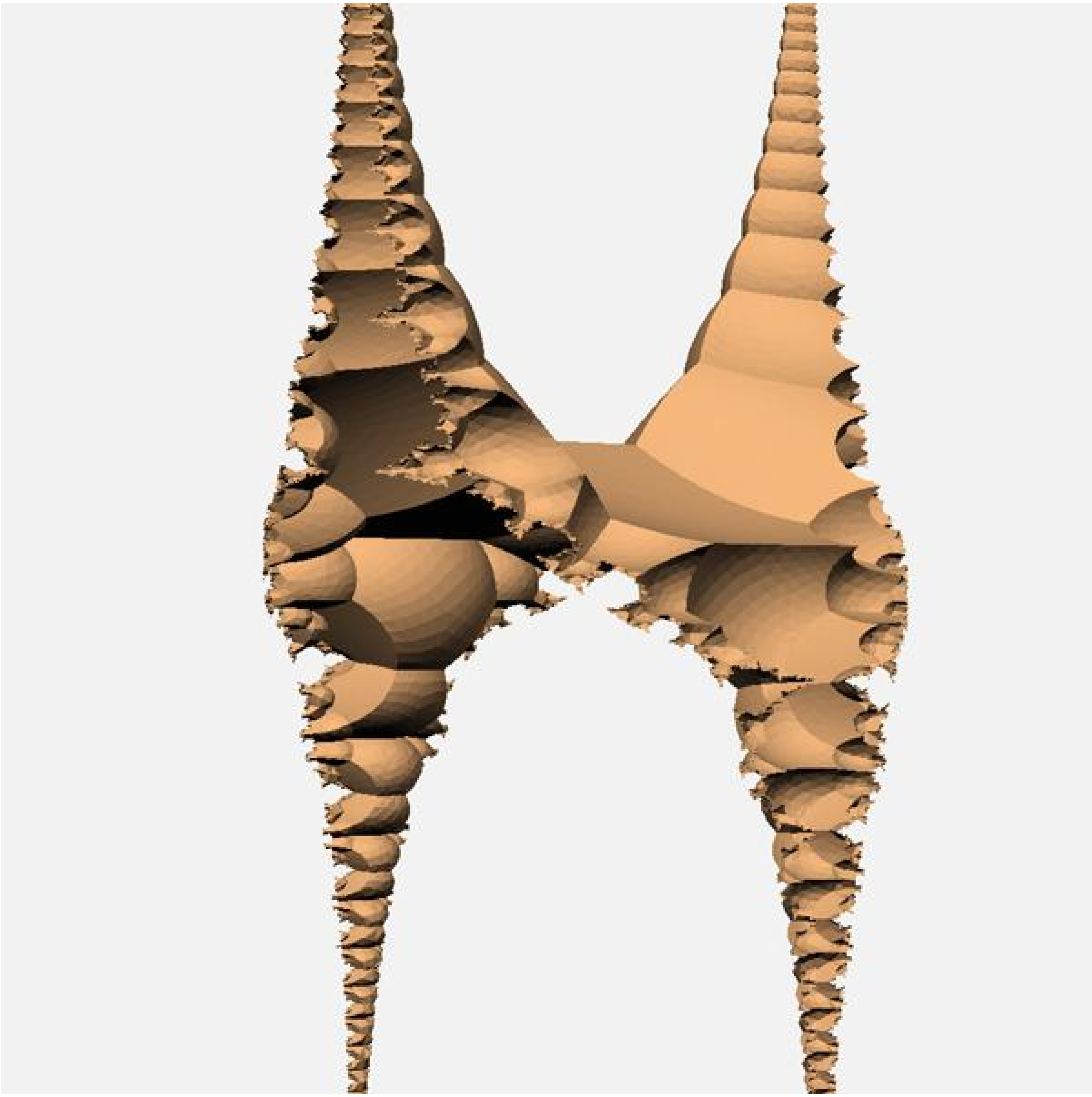
The same tiling in shown in the cylinder model.
Axis of cylinder is aligned with axis of some hyperbolic transformation. It makes tiling periodic.

Samples of 3D hyperbolic tilings



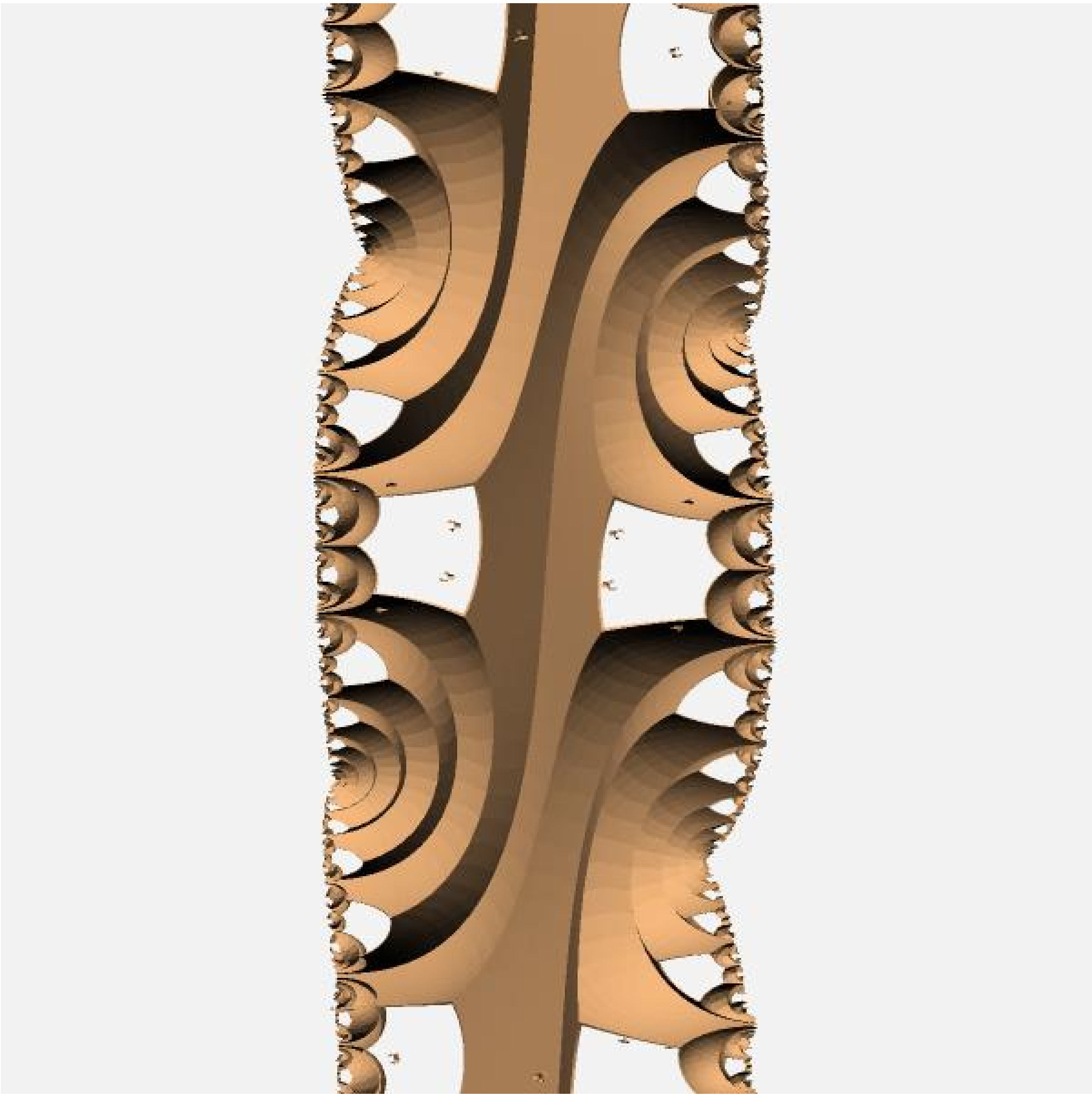
Tiling with two generator free group shown in the cylinder model. Generators have almost parabolic commutator.

Samples of 3D hyperbolic tilings



Tiling with group with parabolic commutator shown in the cylinder model. Fixed points of two parabolic transformations are located at the cylinder's axis at infinity.

Samples of 3D hyperbolic tilings



Some quasifuchsian group with 5 generators shown in the cylinder model.

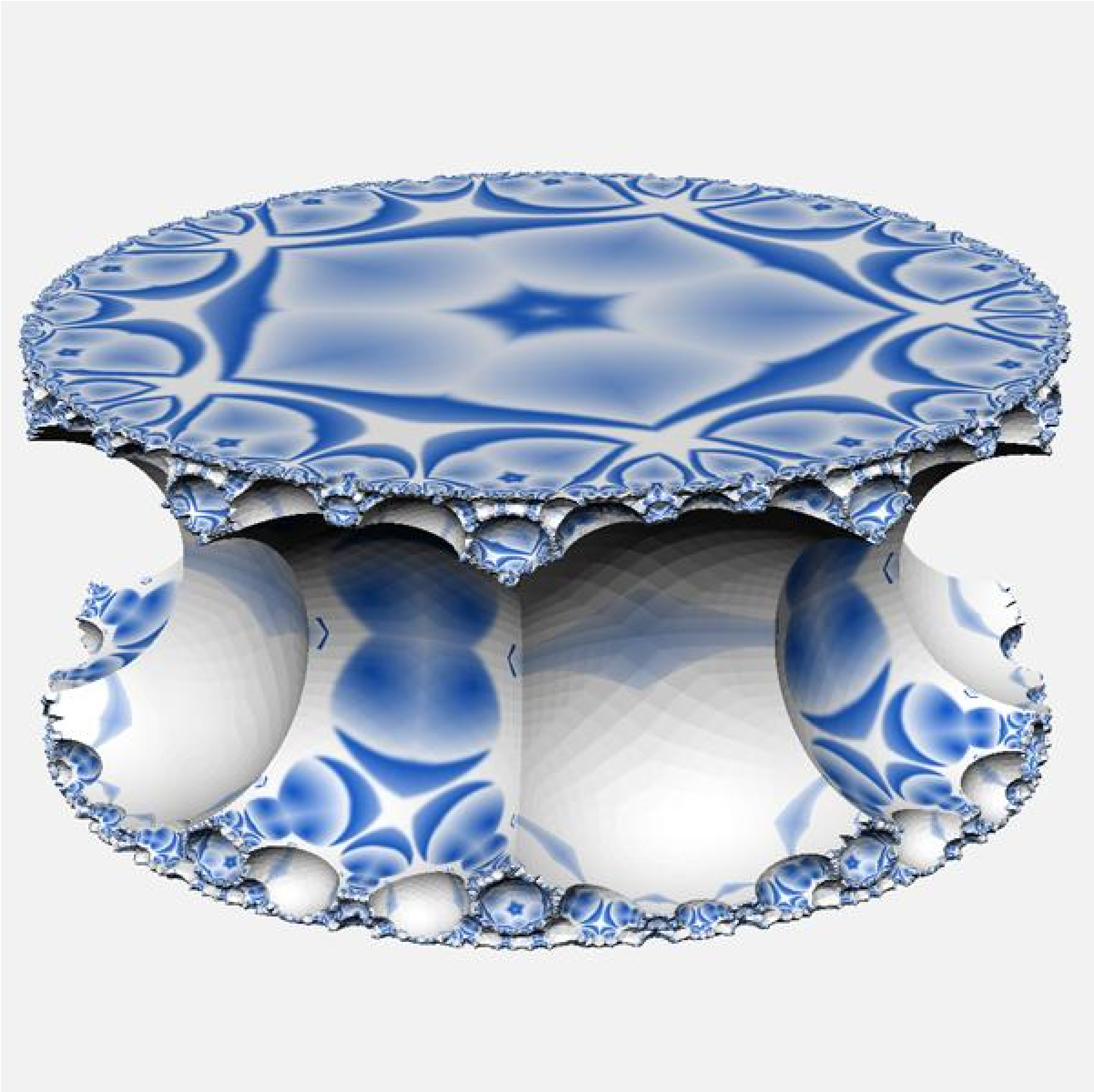
Samples of 3D hyperbolic tilings

Wild quasifuchsian group.

Samples of 3D hyperbolic tilings

Wilder quasifuchsian group.

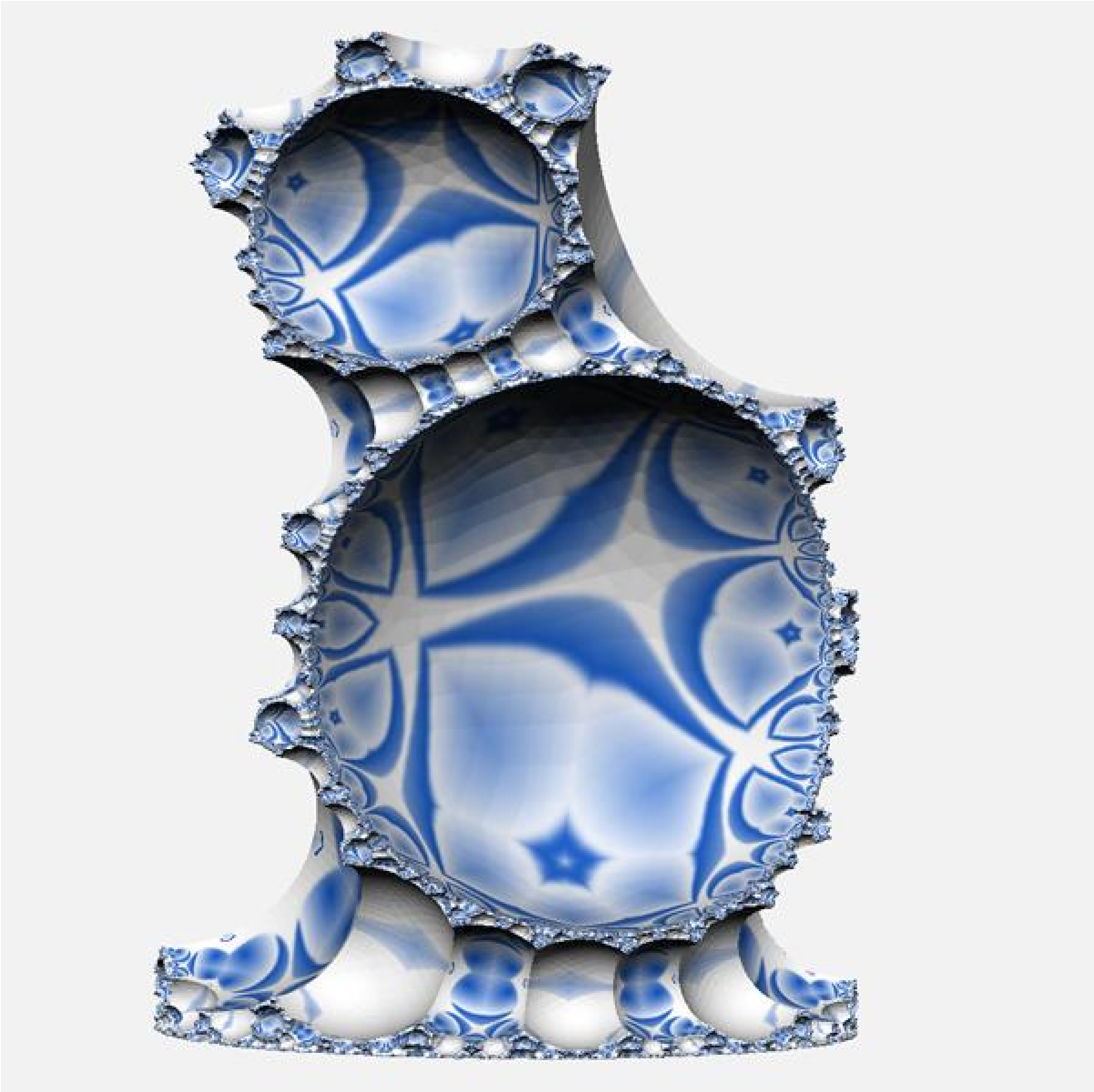
Samples of 3D hyperbolic tilings



Tiling with truncated cubes in the cylinder model.
Axis of the cylinder model is alligned with common perpendicular to 2 planes. This makes both these planes flat.

Placing repeating pattern on the surface reveals the *455 tiling structure on the hyperplanes, which corresponds to $\frac{\pi}{4}, \frac{\pi}{5}, \frac{\pi}{5}$ triangle formed at the truncated vertex.

Samples of 3D hyperbolic tilings



Another orientation the same tiling with cubes in the cylinder model.

The axis of cylinder model is perpendicular to one selected plane, which make this plane at the bottom flat.

Samples of 3D hyperbolic tilings

Another orientation the same tiling.
*455 pattern is repeated at every exposed plane.

Samples of 3D hyperbolic tilings

Another orientation the same tiling.

To find this orientation we selected a loxodromic transformation with small rotational and large translational components. Next we selected 2 hyperbolic planes with poles near the 2 fixed points of the transformation and have alligned common perpendicular to these planes with the axis of the cylinder model.

Samples of 3D hyperbolic tilings

Another orientation of the same tiling.

One plane at the bottom is perpendicular to the cylinder axis. Another plane is almost parallel the the axis, making long tong at the top.

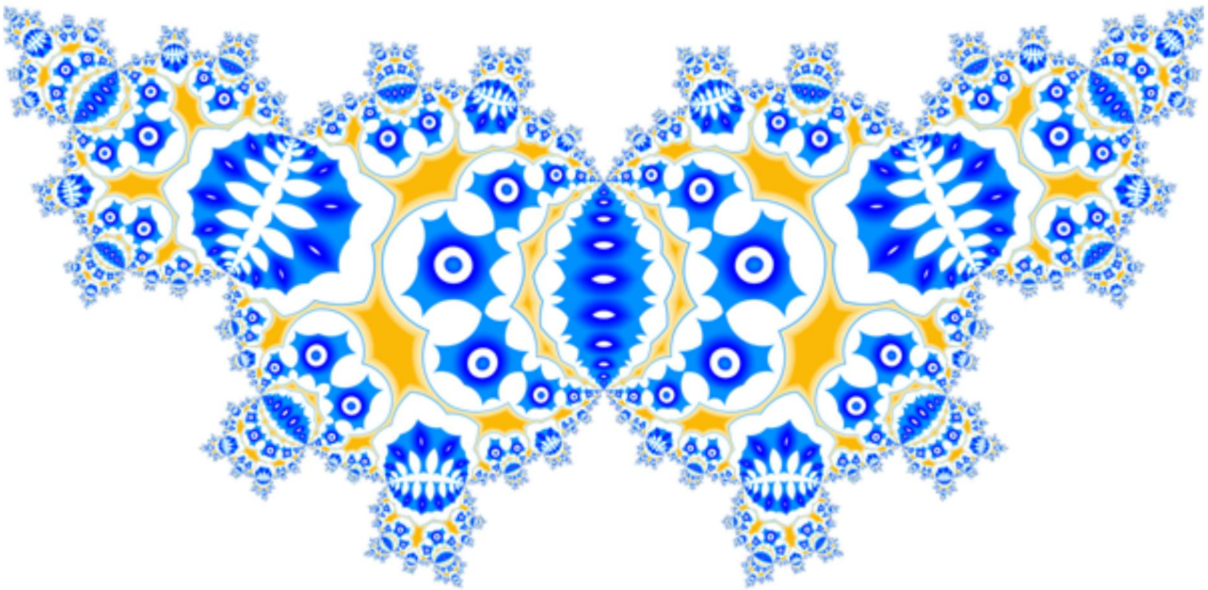
Samples of 3D hyperbolic tilings

Another orientation of the same tiling with cubes in the cylinder model.

One of the hyperplanes is very close to the cylinder's axis and is stretched almost to the vertical band.

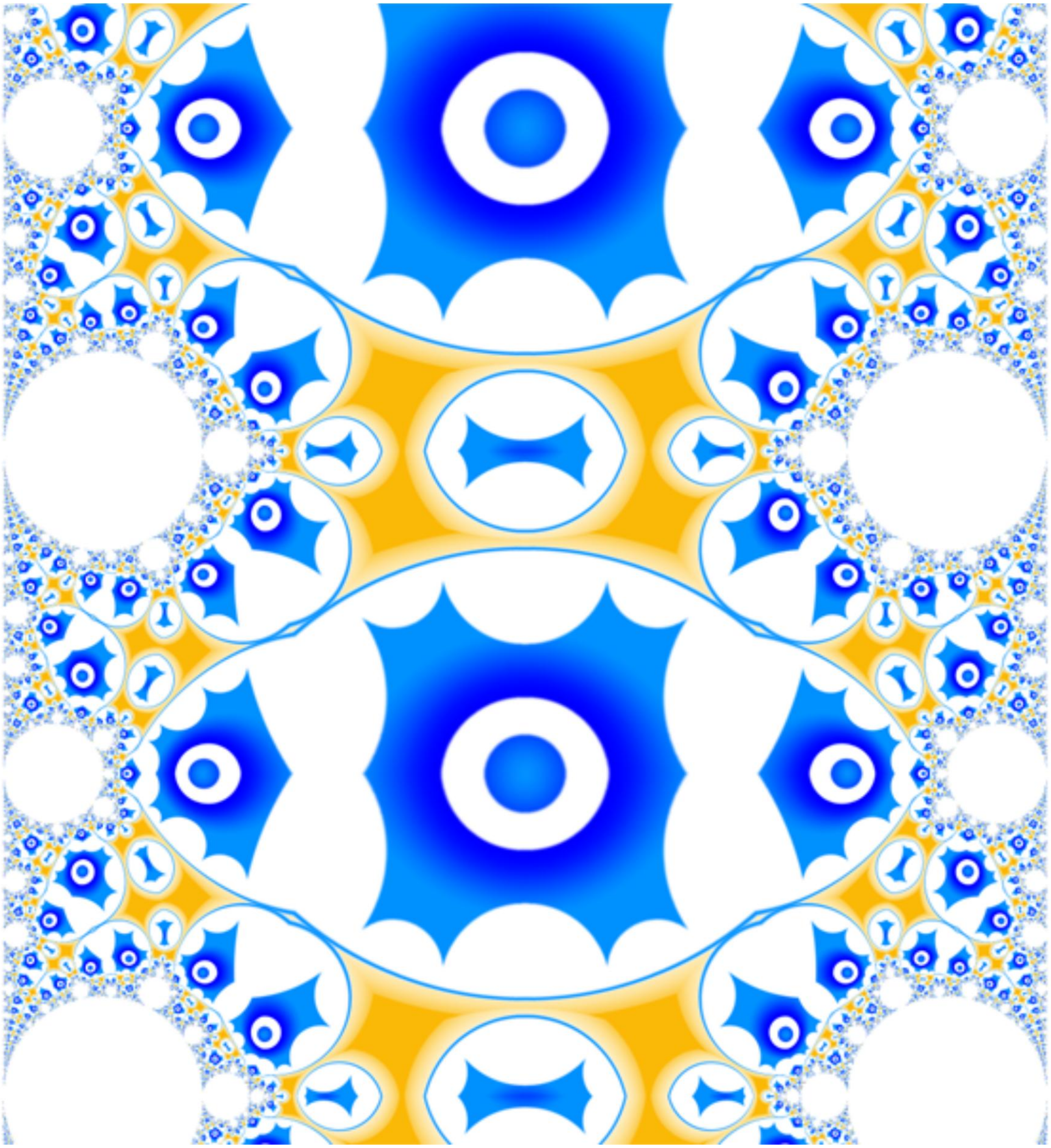
On the top and at the bottom of the tiling there are flat hyperplanes, which have their common perpendicular aligned with the cylinder's axis.

Samples of tiling at infinity



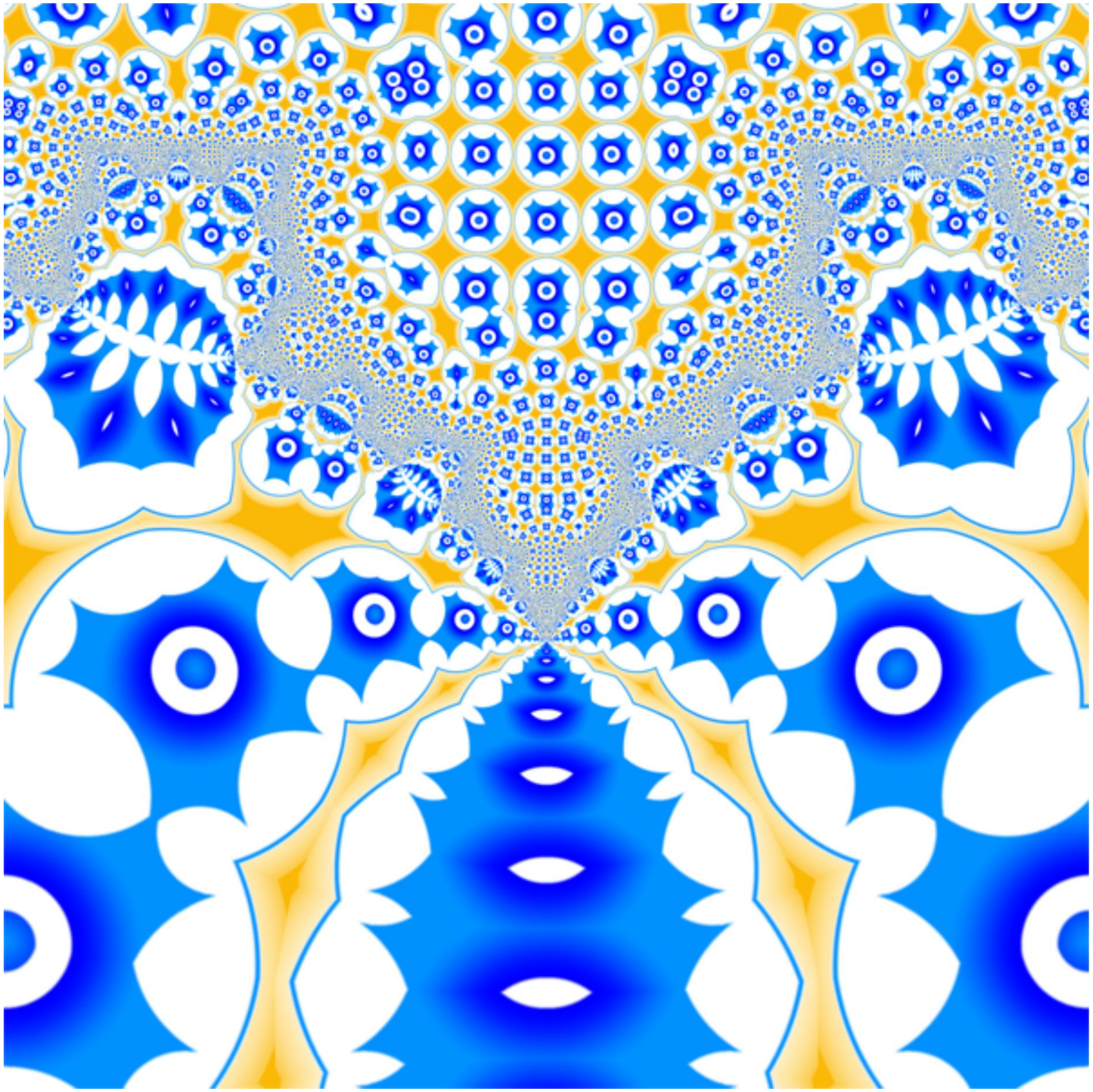
Tiling at the boundary of hyperbolic space shown in stereographic projection to 2D.

Samples of tiling at infinity



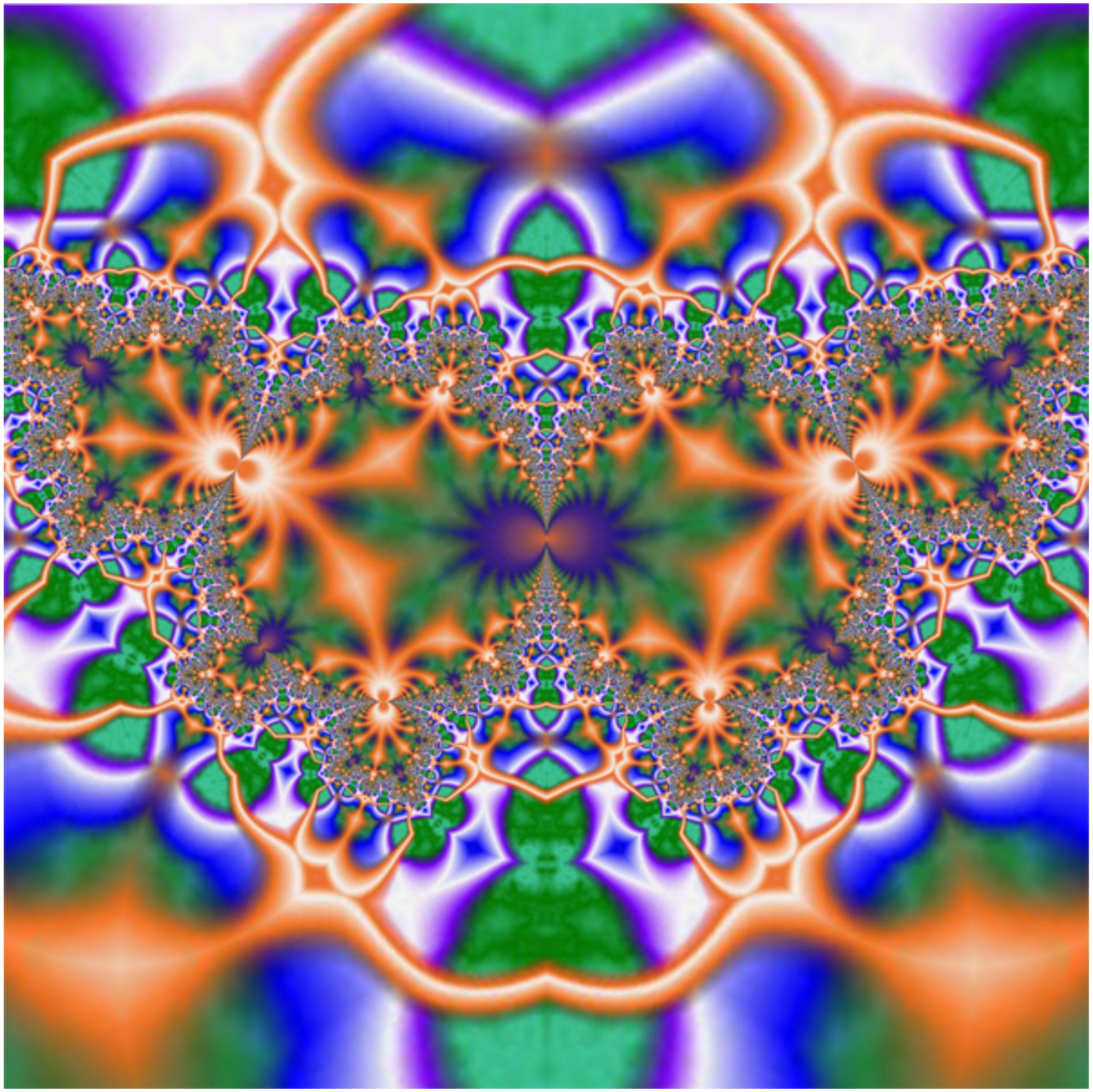
Tiling at the boundary of hyperbolic space shown in stereographic projection to 2D.

Samples of tiling at infinity



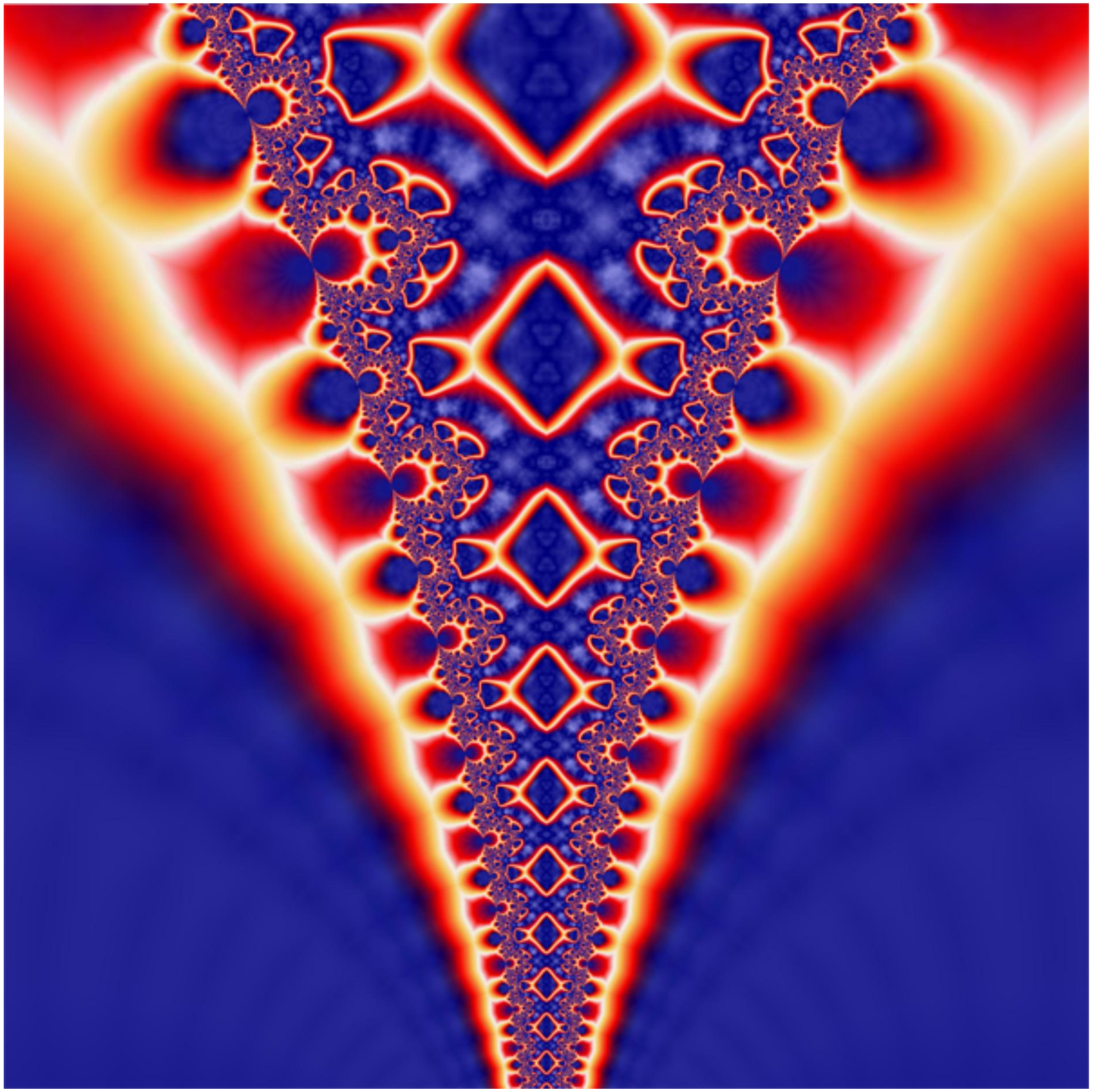
Tiling at the boundary of hyperbolic space shown in stereographic projection to 2D.

Samples of tiling at infinity



Tiling at the boundary of hyperbolic space shown in stereographic projection to 2D.

Samples of tiling at infinity



Tiling at the boundary of hyperbolic space shown in stereographic projection to 2D.

Implementation in Metal

Metal sculpture. 20 Dodecahedra.

Implementation in Metal

Metal sculpture. First iteration of Weber-Seifert dodecahedral tiling.

Implementation in Metal

Metal sculpture. 20 hyperbolic cubes.

Implementation in Metal

Metal sculpture. Tiling by Fuchsian group.

Implementation in Metal

Hyperbolic bracelet.

Implementation in Metal

Hyperbolic bracelet.

Implementation in Metal

Hyperbolic bracelet.

Implementation in Metal

Hyperbolic bracelet.

Implementation in Metal

Hyperbolic pendant.

Implementation in Metal

Hyperbolic pendant.

Implementation in Metal

Hyperbolic pendant.

Hyperbolic Sculpture

Hyperbolic sculpture. Color 3D print.

Hyperbolic Sculpture

Hyperbolic sculpture. Color 3D print.

Hyperbolic Sculpture

Hyperbolic sculpture. Color 3D print.

Hyperbolic Sculpture

Online address of the talk:
bulatov.org/math/1101/